Twisted Drinfeld double : from strings to the Kitaev model

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Quantum gravity in Bordeaux

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Path integral derivation of the operators $T$ that lift the action of a finite group $G$ to the twisted sectors of bosonic strings on the orbifold $\mathcal{M}/G$ in a 3-form magnetic background $H$.

String propagator written as a sum over worldsheets each carrying its own magnetic contribution.

The algebra generated by the operators $T$ is the quasi-quantum group $D_\omega[G]$, introduced in the context of conformal field theory by R. Dijkgraaf, V. Pasquier and P. Roche with

- a product is determined by the commutation with propagation
  \[
  T \begin{array}{c}
  \text{worldsheet}
  \end{array} = \begin{array}{c}
  \text{worldsheet}
  \end{array} T, \tag{1}
  \]

- a coproduct follows from the commutation with the most basic interaction
  \[
  T \begin{array}{c}
  \text{interaction}
  \end{array} = \begin{array}{c}
  \text{interaction}
  \end{array} \Delta T. \tag{2}
  \]

Magnetic amplitude for twisted sectors are also ground states of a version of the Kitaev lattice model.
A **quantum system** is defined by a Hilbert space $\mathcal{H}$ and observables which are Hermitian operators acting on $\mathcal{H}$. A state of the system is defined by a line $s$ in $\mathcal{H}$ (normalized vectors defined up to a phase).

The **probability** of observing the system in the state represented by $\chi$ knowing that it is in the state represented by $\psi$ is $|\langle \psi, \chi \rangle|^2$.

A **symmetry** is a transformation of the space of states $s \rightarrow s'$ preserving the transition probabilities, $|\langle \psi', \chi' \rangle|^2 = |\langle \psi, \chi \rangle|^2$.

**Theorem (Wigner)**

*Each symmetry acting on states $s \rightarrow s'$ can be implemented by a unitary or antiunitary operator $U$ on $\mathcal{H}$.*

$$\psi \in s \Rightarrow \psi' = U\psi \in s'$$

*and these operators are unique up to a phase.*

Antiunitary symmetries: time reversal $T$, charge conjugation $C$
Projective representations

If a group $G$ acts on the states preserving the transition probabilities, the operators $U_g$ are only defined up to phases

$$U_g U_h = \omega_{g,h} U_{gh}$$

Projective representations are classified using group cohomology

- associativity constraint: $\omega$ is a 2-cocycle

$$U_g (U_h U_k) = (U_g U_h) U_k \iff \omega_{h,k} \omega_{g,h,k}^{-1} \omega_{g,h}^{-1} = 1 \quad (\delta \omega)_{g,h,k}$$

- triviality: $\omega$ is a coboundary

$$\omega_{g,h} = \eta_h \eta_{g,h}^{-1} \eta_g \iff V_g V_h = V_{gh} \text{ with } V_g = \eta_g U_g \quad (\delta \eta)_{g,h}$$

General group cohomology: $n$-cochains are functions on $n$ copies of $G$ with values in a abelian group carrying an action of $G$, $\delta^2 = 0$ with

$$\delta \omega(g_0, g_1, \ldots, g_n) = g_0 \cdot \omega(g_1, \ldots, g_n)$$

$$\times \prod_{i=0}^{n-1} \left[ \omega(g_0, \ldots, g_i g_{i+1}, \ldots) \right]^{(-1)^{i-1}} \times \left[ \omega(g_0, g_1, \ldots, g_{n-1}) \right]^{(-1)^{n-1}}$$
Magnetic amplitude for a particle

For a particle on a manifold $\mathcal{M}$ in a magnetic background $B$ (closed 2-form with integral periods), wave functions are sections of a line bundle $\mathcal{L}$ over $\mathcal{M}$ with a connection $\nabla$ of curvature $B$.

In the path integral approach, the kernel of the evolution operator is

$$K(y, x) = \int [D\varphi] \ e^{-S[\varphi]} \mathcal{A}[\varphi],$$

(3)

$S$ classical action (not involving the magnetic field) $\mathcal{A}[\varphi]$ holonomy of $\nabla$ along the path $\varphi$.

Both $\mathcal{L}$ and $\mathcal{A}[\varphi]$ are constructed using a good open cover $\{U_i\}$

$$\begin{cases}
B_i &= dA_i & \text{on} & U_i, \\
A_j - A_i &= i \log f_{ij} & \text{on} & U_i \cap U_j, \\
f_{jk}(f_{ik})^{-1}f_{ij} &= 1 & \text{on} & U_i \cap U_j \cap U_k,
\end{cases}$$

(4)

Invariance under gauge transformations of $A_i$ and $f_{ij}$.
Explicit expression of the magnetic amplitude

Using a cover of the path, the magnetic amplitude (holonomy of the connection along the path) is

\[ A_{ij}[\varphi] = \exp \left\{ \sum_{l_{\alpha} \in I} \int_{l_{\alpha}} \varphi^* A_{i_{\alpha}} \right\} \prod_{\begin{subarray}{l} l_{\alpha} \in I \\ v_{\beta} \in \partial l_{\alpha} \end{subarray}} f_{i_{\alpha}j_{\beta}}^{-\epsilon_{\alpha\beta}}(\varphi(v_{\beta})) , \tag{5} \]

where \( \epsilon_{\alpha\beta} = +1 \) if \( l_{\alpha} \) is arriving at \( v_{\beta} \) and \(-1\) if it is leaving.

In accordance with its interpretation as a map from the fibre at \( x \) of \( \mathcal{L} \) to that at \( y \), it is **independent of the covering** and **gauge invariant**, except at the boundaries.
Projective group action on wave functions

**Classical symmetry**: Action of a (finite) group $G$ on $\mathcal{M}$ such that $S$ is genuinely invariant and $g^* B = B$.

**Quantum symmetry**: Lift of the action of $G$ to the Hilbert space $\mathcal{H}$ of wave functions ($\phi_g$ isomorphism between $(g^* \mathcal{L}, g^* \nabla)$ and $(\mathcal{L}, \nabla)$)

$$T_g \psi(x) = \phi_g(x) \psi(x \cdot g)$$  \hspace{1cm} (6)

The phases are determined in the path integral formalism by the **commutation** of $T_g$ with propagation ($K T_g = T_g K$)

$$\mathcal{A}[\varphi \cdot g] = \phi_g^{-1}(y) \mathcal{A}[\varphi] \phi_g(x).$$  \hspace{1cm} (7)

**Projective representation** $T_g T_h = \omega_{g,h} T_{gh}$ with the group 2-cocycle

$$\omega_{g,h} = \phi_h(x \cdot g) \phi_{gh}^{-1}(x) \phi_g(x).$$  \hspace{1cm} (8)

The operators $T_g$ generate the **twisted group algebra**.

The cohomology class of $\omega$ is an **obstruction** to the existence of a quantum theory on $\mathcal{M}/G$ (no invariant states in $\mathcal{H}$).

Generalization of **magnetic translations** for a particle on $\mathbb{R}^N$ in a uniform magnetic field with $G = \mathbb{Z}^N$. (twisted group algebra = noncommutative torus.)
Magnetic fields for closed strings

A closed string on \( \mathcal{H} \) sweeping a \textbf{worldsheet} \( \Sigma \) couples to a 2-form magnetic potential \( B \) (Kalb-Ramond field) with 3-form field strength \( H = dB \)

\[
\int_\Sigma B \quad \rightarrow \quad e^{i \int_\Sigma B}
\]  

(9)

In general the potentials are only locally defined and correspond to a \textbf{gerb with connection} from which we compute the holonomy around \( \Sigma \) using a triangulation

\[
\begin{align*}
H_i &= dB_i \quad \text{on} \quad U_i, \\
B_j - B_i &= dB_{ij} \quad \text{on} \quad U_i \cap U_j, \\
B_{jk} - B_{ik} + B_{ij} &= i \, d \log f_{ijk} \quad \text{on} \quad U_i \cap U_j \cap U_k, \\
f_{jkl}(f_{ikl})^{-1}f_{ijl}(f_{ijk})^{-1} &= 1 \quad \text{on} \quad U_i \cap U_j \cap U_k \cap U_l,
\end{align*}
\]  

(10)

with two layers of gauge transformations.

Example of \textbf{WZW} models with \( \mathcal{M} = SU(N) \) and \( H = \frac{k}{12\pi} \text{Tr}(g^{-1}dg)^3 \).

Interpretation of the holonomy around cylinders as parallel transport for a line bundle over the \textbf{loop space}. 
Tricomplex with de Rham, Čech and group cohomologies

**Tricomplex** with cochains $C_{p,q,r}$ that are de Rham forms of degree $p$, defined on $(q + 1)$-fold intersections of a "good invariant cover", $U_{i_0} \cap \cdots \cap U_{i_q}$ and functions of $r$ group indices.

Three **commuting differentials**

- de Rham differential in the $p$ direction (idlog for functions)
- Čech coboundary $\check{\delta}$ in the $q$ direction
- group coboundary $\delta$ in the $r$ direction

For any fixed value of $r$, we have a Čech-de Rham bicomplex,

$$C_{r,s}^{\text{tot}} = \bigoplus_{p+q=s} C_{p,q,r},$$

with the **Deligne differential** defined by $D = \mp d \pm \check{\delta}$ fulfilling $D^2 = 0$ and $\delta D = D\delta$. 
Symmetries of 2-form potentials

Starting with \( H = (H_i, 0, 0, 1) \in C_{0,3}^{\text{tot}} \) such that \( \mathcal{D}H = 0 \) and \( \delta H = 0 \) (globally defined closed invariant 3-form), we solve a series of cohomological equations ending in a constant 3-cocycle \( \omega \in C_{3,0}^{\text{tot}} \), with gauge ambiguities in the definition of \( B \) and \( A \).

\[ g^* B - B = \mathcal{D}A_g \]
\[ g^* A_h - A_{gh} + A_g = \mathcal{D}\Phi_{g,h} \]
\[ g^* \Phi_{h,k} (\Phi_{gh,k})^{-1} \Phi_{g,hk} (\Phi_{g,h})^{-1} = \omega_{g,h,k} \]
Magnetic amplitude for twisted sectors

**Twisted sectors** on $\mathcal{M}/G$ are strings $X : [0, 2\pi] \to \mathcal{M}$ that close up to their winding $w \in G : \quad X(2\pi) = X(0) \cdot w$.

Free string propagation involves a path integral with **magnetic amplitude**

$$A[\phi] = e^{i \int_\Sigma B + i \int_x A_w}$$

for the cylinder with **cut** and **triangulation** embedded in $\mathcal{M}$

String wave functions $\Psi = \text{sections of a line bundle over twisted sectors}$

Magnetic amplitude for cylinders = parallel transport

Invariance under simultaneous **gauge transformations** of $B$, $A$, $\Phi$ and $\Psi$
Stringy magnetic translations and their algebra

Stringy magnetic translations $T^w_g : \mathcal{H}_{wg} \to \mathcal{H}_w$ lift the group action to the twisted sectors commuting with propagation

$$T^w_g \psi(X) = \Gamma_{w,g}(x) e^{-i \int_x^{xw} A_g \psi(X \cdot g)},$$

with $\Gamma_{w,g} = \Phi_{g,wg} \Phi_{w,g}^{-1}$ and $wg^g = g^{-1}wg$.

Projective representation on the twisted sectors identical to the multiplication law of the quasiquantum group $D_\omega[G]$

$$T^w_g T^v_h = \delta_{v,wg} \frac{\omega_w,g,h \omega_{g,h,wg}}{\omega_{g,wg,h}} T^w_{gh}.$$ (14)

Combinatorial interpretation in terms of tetrahedra representing the 3-cocycle (transgression $(n+1)$-cocycle $\to n$-cocycle depending on $w$)
Interactions

Most basic interaction involves pair of pants

\[ A[\varphi] = e^{i \int_{\Sigma} B + i \int_{t}^{x} A_{vw} + i \int_{t}^{y} A_{v} + i \int_{t}^{z} A_{w} \Phi_{v,w}^{-1}(t)} \]

(15)

\( \Phi \) is inserted at the splitting point to maintain gauge invariance for \( A \).

Global anomalies for magnetic amplitudes on arbitrary surfaces that depend on \( \omega \) and on a representation of \( \pi_1(\Sigma) \).

Consistency condition \( \omega = 1 \) for the orbifold \( M/G \). (Analogous to the particle’s case.)
Quasi-Hopf algebras

An algebra $\mathcal{A}$ is a Hopf algebra if it admits a counit $\epsilon : \mathcal{A} \to \mathbb{K}$, coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and an antipode $S : \mathcal{A} \to \mathcal{A}$ such that $m \circ (S \otimes \text{id}) \circ \Delta = m \circ (\text{id} \otimes S) \circ \Delta = \epsilon$.

Examples:
- group algebra $\mathbb{C}[G] = \{ \sum a(g) g \}$ with $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$ and $S(g) = g^{-1}$
- functions on $G$ with pointwise product, $\Delta f(g, h) = f(gh)$, $\epsilon(f) = 1$, $Sf(g) = f(g^{-1})$.

Modules over a Hopf algebra form a category with trivial representation ($\epsilon$), tensor products ($\Delta$) and duals ($S$).

A quasi-Hopf algebra $\mathcal{A}$ has a coproduct associative up to the Drinfel’d associator $\Omega \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$

$$(\text{id} \otimes \Delta) \circ \Delta = \Omega [(\Delta \otimes \text{id}) \circ \Delta] \Omega^{-1}$$

obeying the pentagon axiom.

A bialgebra is quasi-cocommutative, if there is an invertible $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ $\Delta^{\text{op}}(b) = \mathcal{R} \Delta(b) \mathcal{R}^{-1}$

$\Rightarrow$ braid group action $\sigma \circ R : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_2 \otimes \mathcal{H}_1$, $\sigma(\psi \otimes \chi) = \chi \otimes \psi$
Derivation of the coproduct

Commutation of the orbifold group action with the decay process dictates the action of $T^w_g$ on $\mathcal{H}_u \otimes \mathcal{H}_v$ form which we read the coproduct

$$\Delta(T^u_g) = \sum_{vw=u} \frac{\omega_v,w,g \omega_g,v^g,w^g}{\omega_v,g,w^g} T^v_g \otimes T^w_g. \quad (16)$$

Combinatorial interpretation of the extra phase in the action on tensor products

The operators $T^w_g$ generate the quasi-quantum group $D_\omega[G]$ which is a quasi-triangular quasi-Hopf algebra deformation of the quantum double of the group algebra of $G$. 
Action of the quasi-Hopf algebra

$\mathcal{R}$-matrix defines a **braid group** action on tensor products.

Coassociativity up to the **Drinfeld associator**: states in $(\mathcal{H}_u \otimes \mathcal{H}_v) \otimes \mathcal{H}_w$ and in $\mathcal{H}_u \otimes (\mathcal{H}_v \otimes \mathcal{H}_w)$ only differ by the global phase $\omega_{u,v,w}$.

**Antipode** related to reversing the string orientation $S(T_w^g) = \propto T_{g^{-1}}^{w^{-1}}$. 
Discrete de Rham cohomology

A $n$-form $\Omega(x_0, \ldots, x_n)$ is an antisymmetric function on $X^{n+1}$ with values in an abelian group, equipped with a differential

$$d\Omega(x_0, \ldots, x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \Omega(x_0, \ldots, \hat{x}_i, \ldots, x_{n+1})$$

where $\hat{x}_i$ denotes the $i$-th coordinate removed.

Geometrical interpretation: $\Omega(x_0, \ldots, x_n)$ flux over a $n$-simplex with $n + 1$ vertices $x_0, \ldots, x_n$.

Some simple examples

$$d\Phi(x, y) = \Phi(y) - \Phi(x)$$
$$dA(x, y, z) = A(y, z) - A(x, z) + A(x, y)$$
$$dB(x, y, z, t) = B(y, z, t) - B(x, z, t) + B(x, y, t) - B(x, y, z)$$

This differential is nilpotent $d^2 = 0$. 
Kitaev model

Kitaev model defined on a triangular graph $\Gamma$ on a surface $\Sigma$ with Hilbert space constructed by assigning group elements to the (oriented) edges

$$\Psi(\{g_e\}) \in \mathcal{H} = \bigotimes_{\text{edges}} \text{Fun}(G \to \mathbb{C})$$

(17)

$$H = - \sum_{\text{faces } f} P_f - \sum_{\text{vertices } v} \delta_v$$

(18)

$P_f$ (translation of face variables $u \to g^{-1}u, w \to wg, \ldots$) and $\delta_v$ (constraint $uvw^{-1} = 1$) are mutually commuting projectors

Ground states given by moduli space $\text{Hom}(\pi_1(\Sigma) \to G)/\text{Ad}(G)$
Twisted Kitaev model

Triangulate each face of $\Gamma$ and decorate vertices with variables $x_v \in X$

$$\Psi(\{g_e\}, \{x_v\}) \in \bigotimes_{\text{edges}} \text{Fun}(G \to \mathbb{C}) \bigotimes_{\text{vertices}} \text{Fun}(X \to \mathbb{C})$$

$$P_f \psi(x, w, \ldots) = \psi(xg, g^{-1}w, \ldots) \times \prod_{\text{vertices in } \partial f} \omega_{u,v,g} \quad \text{for } x, w \in f$$

$$\delta_v = \prod_{\text{around } v} \delta_{x_j, x_i g_{ij}}$$
Ground states

A ground state can be constructed using the previous gerbe amplitude

\[ \psi(\{g_e\}, \{x_v\}) = \prod_{\text{vertices}} \delta_{x_u,y} \delta_{y_v,z} \delta_{y_w,z} \times \]
\[ \times \prod_{\text{triangles}} \exp iB(x, y, z) \prod_{\text{edges}} \exp iA_w(x, y) \prod_{\text{vertices}} \Phi_{u,v}(x) \]

- lift of a single corner

\[ \Phi_{g^{-1}u,u^{-1}v}(xg) \omega_{g,g^{-1}u,u^{-1}v} = \frac{\Phi_{g,g^{-1}v}(x)}{\Phi_{u,u^{-1}v}(x) \Phi_{g,g^{-1}u}(x)} \]

- lift of a triangle:

\[ B(xg, yg, zg) = B(x, y, z) + A_g(x, y) + A_g(y, z) + A_g(z, x) \]

- lift of an edge

\[ A_{g^{-1}w(xg, yg)} = A_w(x, y) - A_g(x, y) + \log -i\Phi_{g,g^{-1}w(y)} + \log -i\Phi_{g,g^{-1}w(x)} \]
Conclusion and outlooks

$D_\omega[G]$ is a higher dimensional generalization of projective group representations

<table>
<thead>
<tr>
<th>particles</th>
<th>strings</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-form $B$</td>
<td>3-form $H$</td>
</tr>
<tr>
<td>line bundle</td>
<td>gerbe</td>
</tr>
<tr>
<td>2-cocycle $\omega$</td>
<td>3-cocycle $\omega$</td>
</tr>
<tr>
<td>twisted group algebra</td>
<td>quasi-quantum group</td>
</tr>
</tbody>
</table>

In both case, the theory on $\mathcal{M}/G$ is **consistent** only if $\omega = 1$.

**Application:** **Discrete torsion for open membranes** (work in progress)

Discrete torsion for **open strings** induces a projective group action on the wave functions of the endpoints (M. Douglas).

**M-theory** discrete torsion is a 3-cocycle $\alpha$ (E. Sharpe).

**finite group** Chern-Simons theory (R. Dijkgraaf and E. Witten) with gauge group $G$ and action derived from $\alpha$ provides a way to assign phases to the **worldvolume** swept by the membrane.

$D_\alpha[G]$ acts on the wave functions of the endlines of the membranes (work in progress).