Duality of Orthogonal and Symplectic Random Tensor Models

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Joint work with H. Keppler, T. Krajewski, A. Tanasa

Based on : Keppler & Muller, arXiv :2304.03625

Keppler, Krajewski, Muller, Tanasa arXiv :2307.01527

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Introduction and motivation

Tensor models consist of tensors whose components are random variables given by the distribution :

$$f(T) = e^{-S[T]}, \quad S[T] = T^{a_1 \dots a_D} C_{a_1 \dots a_D b_1 \dots b_D} T^{b_1 \dots b_D} - \sum_{\mathcal{B}} \lambda_{\mathcal{B}} I_{\mathcal{B}}(T).$$

The action is invariant under a group G.

$$Z = \int [dT] f(t), \quad \langle P(T) \rangle = \frac{1}{Z} \int [dT] f(T) P(T)$$

Why do we study tensor models ?

- natural generalization of matrix models
- approach to random geometry in dimension 3 and higher

The relation $O(-N) \approx Sp(N)$

Occurence of this relation :

- Representation theory (Cvitanovic & Kennedy, 1982)
- Vector models (LeClerc and Neubert, 2007)
- $SO(N)$ and $Sp(N)$ gauge theories (Mkrtchyan, 1981)
- Orthogonal and Symplectic random matrix models (Mulase and Waldron, 2003)

In Gurau & Keppler, arXiv :2207.01993, AIHPD (in press) this relation was extended to tensor models with quartic interaction (see Keppler VTJC talk)

How to prove it for more general tensor models ?

Objective of the talk

- Extend the duality to random tensor models with :
  - General interactions Keppler & Muller, arXiv :2304.03625, LMP (in press)
  - Irreducible representations of $O(N)$ and $Sp(N)$ Keppler, Krajewski, Tanasa, Muller arXiv :2307.01527

Main result

Graded tensor model = no symmetry of the indices of T

Symmetric graded tensor model = symmetry of the indices of T

Symmetry in 2^{nd} model given by Young tableau λ.
Grading allows to study the $O(N)$ and the $Sp(N)$ case simultaneously

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Amplitudes of graphs of the two models :

$$K(\{\lambda\},\mathcal{G})\cdot\prod_{c\in\mathcal{D}}\left((-1)^{|c|}N_c\right)^{F_{c/\mathbf{0}}(\mathcal{G})}\text{ for the first model}$$

$$K(\{\lambda\},\mathcal{G})\cdot\left((-1)^{\mathfrak{b}}N\right)^{F(\mathcal{G})}\text{ for the second model}$$

$|c|$ and $\mathfrak{b}$ grading parameters controlling the symmetry ($O(N)$ or $Sp(N)$).

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$|c|$ and b grading parameters controlling the symmetry ($O(N)$ or $Sp(N)$).
Factors $(-1)^{|c|}N_c$ or $(-1)^{b}N$ $\rightarrow$ Models invariant under :

$$|c| \rightarrow |c|+1 \text{ and } N_c \rightarrow -N_c\,, \quad 1^{st} \text{ model}$$

$$\mathfrak{b} \rightarrow \mathfrak{b}+1 \text{ and } N \rightarrow -N\,, \quad 2^{nd} \text{ model}$$

Invariance of first model :

Tensor models with no symmetry properties under permutation of indices and invariance under tensor product of different copies of $O(N)$ or $Sp(N)$ are dual to corresponding tensor models obtained by changing the symmetry of a given color.

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Tensor models with symmetry given by the $O(N)$ irreducible representation R are dual to corresponding tensor models with $Sp(N)$ symmetry given by the representation with transposed Young diagrams R' (exchanging symmetrization and antisymmetrization).

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For both models, the duality is said to hold in the sense that the amplitudes of graphs in their perturbative expansions are mapped into each other after a change of N to $-N$.

- Sign of directed pairings
- Graded tensor model :
  - The model, colored graphs and invariants
  - Invariance of the model
- Symmetric graded model :
  - Brauer algebra and the traceless projector
  - The model, stranded graphs and invariants
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- Conclusion

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Sign of directed pairings

Definition : Let $\vec{M}_1$ and $\vec{M}_2$ be two directed pairings on a set of $2D$ elements given by

$$\vec{M}_1 = \{(i_1, i_2), \ldots, (i_{2D-1}, i_{2D})\} ,$$
$$\vec{M}_2 = \{(j_1, j_2), \ldots, (j_{2D-1}, j_{2D})\} .$$

The sign of $\vec{M}_1$ with respect to $\vec{M}_2$ is defined as

$$\epsilon(\vec{P}_1, \vec{P}_2) = \mathrm{sgn}\left(\begin{pmatrix} i_1 & i_2 & \cdots & i_{2D-1} & i_{2D} \\ j_1 & j_2 & \cdots & j_{2D-1} & j_{2D} \end{pmatrix} \right)$$

Properties :

- Symmetric under permutation of its arguments :

$$\epsilon(\vec{M}_1, \vec{M}_2) = \epsilon(\vec{M}_2, \vec{M}_1) .$$

- For three pairings $\vec{M}_1$, $\vec{M}_2$, $\vec{M}_3$ on the same set :

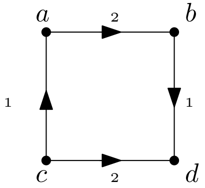
$$\epsilon(\vec{M}_1, \vec{M}_2) = \epsilon(\vec{M}_1, \vec{M}_3)\epsilon(\vec{M}_2, \vec{M}_3) .$$

Diagrammatic representation of pairings

D pairings on a set S $\to$ directed D-colored graph.

Depict :

- The elements of S as nodes
- Each pair of $\vec{M}_i$ (on the set) as oriented edge (pointing from 1^{st} element to 2^{nd}) of color i.



$$S = \{a, b, c, d\}$$

$$\vec{M}_1 = \{(c, a), (b, d)\}$$
$$\vec{M}_2 = \{(a, b), (c, d)\}$$

Sign $\epsilon(\vec{M}_i, \vec{M}_j)$:

$$\epsilon(\vec{M}_i, \vec{M}_j) = (-1)^{F_{i/j,even}}$$

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The tensors of the model

Tensor $T^{a_1 \dots a_D}$ with no symmetry properties under permutation of indices.
The color $c = 1, \dots, D$ labels the position of the indices.
To each color c is assigned a grading parameter $|c| \in \{0, 1\}$ s.t the model obeys the symmetry :

$$\boldsymbol{O}_1(N_1) \otimes \boldsymbol{O}_2(N_2) \otimes \cdots \otimes \boldsymbol{O}_D(N_D), \quad \boldsymbol{O}_c(N_c) = \begin{cases} O(N_c), & |c| = 0 \\ Sp(N_c), & |c| = 1 \end{cases}$$

Indices of position c are contracted with the bilinear form $g^{|c|}_{ab}$ s.t

$$g^{|c|}_{ab} = \begin{cases} \delta_{ab}, |c| = 0 \\ \omega_{ab}, |c| = 1 \end{cases}, \ \delta = \left(\begin{array}{c|c} \mathbb{1}_{N/2} & 0 \\ \hline 0 & \mathbb{1}_{N/2} \end{array} \right), \ \omega = \left(\begin{array}{c|c} 0 & \mathbb{1}_{N/2} \\ \hline -\mathbb{1}_{N/2} & 0 \end{array} \right) .$$

The tensor components are fermionic if $\sum_c |c|$ odd :

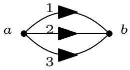
$$T^{a_1 \dots a_D} T^{b_1 \dots b_D} = (-1)^{\sum_c |c|} T^{b_1 \dots b_D} T^{a_1 \dots a_D}$$

Invariants and colored graphs

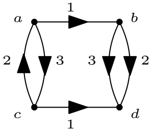
Invariants are built by contracting indices of the tensors with $g^{|c|}_{ab}$.
There is a unique quadratic invariant ($T^{a_{\mathcal{D}}} = T^{a_1 \dots a_D}$)

$$g^{\otimes D}(T, T) = T^{a_{\mathcal{D}}} T^{b_{\mathcal{D}}} \prod_{c \in \mathcal{D}} g^{c}_{a_c b_c} \ .$$

Any invariant can be represented by a directed D-colored graphs $\vec{\mathcal{B}}$ given by c directed pairings $\vec{E}^c(\vec{\mathcal{B}})$ (on the tensors of the invariants).



$$\vec{E}^1 = \{(a,b)\} \quad \vec{E}^2 = \{(a,b)\} \quad \vec{E}^3 = \{(a,b)\}$$



$$\vec{E}^1 = \{(a,b),(c,d)\} \quad \vec{E}^2 = \{(c,a),(b,d)\} \quad \vec{E}^3 = \{(a,c),(b,d)\}$$

$$I_{\vec{\mathcal{B}}}(T) = \sum_{a^1_{\mathcal{D}}, a^2_{\mathcal{D}}, \dots, a^{2k}_{\mathcal{D}}} \Big(\prod_{(i,j) \in \vec{P}_{ref,2k}} T^{a^i_{\mathcal{D}}} T^{a^j_{\mathcal{D}}} \Big) \Big(\prod_{c \in \mathcal{D}} \epsilon(\vec{P}_{ref,2k}, \vec{E}^c(\vec{\mathcal{B}}))^{|c|} \, K^c_{\vec{\mathcal{B}}, a^1_c, \dots, a^{2k}_c} \Big)$$

Class function

We want to only consider independent invariants.

$$I_{\vec{\mathcal{B}}}(T) = \sum_{a^{\mathbf{1}}_{\mathcal{D}}, a^{\mathbf{2}}_{\mathcal{D}}, \dots, a^{\mathbf{2}k}_{\mathcal{D}}} \Bigg(\prod_{(i,j)\in \vec{P}_{ref,\mathbf{2}k}} T^{a^i_{\mathcal{D}}} T^{a^j_{\mathcal{D}}} \Bigg) \Bigg(\prod_{c\in\mathcal{D}} \epsilon(\vec{P}_{ref,2k}, \vec{E}^c(\vec{\mathcal{B}}))^{|c|} \, K^c_{\vec{\mathcal{B}}, a^{\mathbf{1}}_c, \dots, a^{\mathbf{2}k}_c} \Bigg)$$

Sign prescription $\epsilon(\vec{P}_{ref,2k}, \vec{E}^c(\vec{\mathcal{B}}))$ ensures that $I_{\vec{\mathcal{B}}}(T) = I_{\vec{\mathcal{A}}}(T)$ if $\vec{\mathcal{A}}$ and $\vec{\mathcal{B}}$ are two oriented version of the same undirected graph $\mathcal{G}$.

We view an undirected graph as an equivalence class of its directed versions.

We say that $I_{\vec{\mathcal{B}}}(T)$ is a class function.

In the action, we sum over undirected graphs and choose a single representative for each of them.

Definition of the model

The *graded tensor model* is defined by the measure :

$$d\mu[T] = e^{-S[T]}\,[dT], \quad S[T] = \frac{1}{2} g^{\otimes D}(T,T) + \sum_{\substack{\mathcal{B}\ \text{connected},\\ |V(\mathcal{B})|>2}} \frac{\lambda_{\mathcal{B}}}{|V(\mathcal{B})|} I_{\vec{\mathcal{B}}}(T)\,,$$

$$\text{with} \quad [dT] = \prod_{a_{\mathcal{D}}} dT^{a_1 \cdots a_D} \cdot \begin{cases} \frac{1}{(2\pi)^{\prod_c N_c/2}}\,, & \sum_{c=1}^{D} |c| = 0 \mod 2 \\ 1\,, & \sum_{c=1}^{D} |c| = 1 \mod 2 \end{cases},$$

$$Z = \int d\mu[T]\,, \quad \langle I_{\vec{\mathcal{B}}}(T) \rangle = \frac{1}{Z} \int d\mu[T] I_{\vec{\mathcal{B}}}(T)$$

To prove duality, show Z and $\langle I_{\vec{\mathcal{B}}}(T) \rangle$ invariant under

$$|c| \to |c| + 1 \text{ and } N_c \to -N_c$$

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Expansion of the partition function

The partition function can be expanded as :

$$Z = \int [dT] e^{-\frac{1}{2} g^{\otimes}(T,T)} \sum_{\{p_{\mathcal{B}} \geq 0\}} \prod_{\mathcal{B}} \frac{1}{p_{\mathcal{B}}!} \Big(\frac{\lambda_{\mathcal{B}}}{|V(\mathcal{B})|} I_{\mathcal{B}}(T) \Big)^{p_{\mathcal{B}}}$$

Commuting the sum and the integral :

$$Z = \sum_{\{p_{\mathcal{B}} \geq 0\}} \left(\prod_{\mathcal{B}} \frac{1}{p_{\mathcal{B}}!} \Big(\frac{\lambda_{\mathcal{B}}}{|V(\mathcal{B})|} \Big)^{p_{\mathcal{B}}} \right) \langle \prod_{\mathcal{B}} \left(I_{\mathcal{B}}(T) \right)^{p_{\mathcal{B}}} \rangle_0$$

$\prod_{\vec{\mathcal{B}}} \left(I_{\mathcal{B}}(T) \right)^{p_{\mathcal{B}}}$ can be rewritten as a single disconnected invariant $I_{\vec{\mathcal{B}}_{tot}}(T)$

Invariance of $Z \leftrightarrow$ invariance of $\langle I_{\vec{\mathcal{B}}} \rangle_0$

Wick's theorem of the model

The commutation rules of the tensor components is :

$$T^{a_{\mathcal{D}}}\, T^{b_{\mathcal{D}}} = (-1)^{\sum_c |c|}\, T^{b_{\mathcal{D}}}\, T^{a_{\mathcal{D}}}$$

Wick's theorem $(g^{a_{\mathcal{D}} b_{\mathcal{D}}} = \prod_{c \in \mathcal{D}} g^{c,\, a_c b_c})$:

$$\langle T^{a^{\mathbf{1}}_{\mathcal{D}}} \ldots T^{a^{\mathbf{2}k}_{\mathcal{D}}} \rangle_0 = \sum_{P \in \mathcal{P}_{\mathbf{2}k}} \epsilon(\vec{P}_{ref\,,2k}, \vec{P})^{\sum_{c \in \mathcal{D}} |c|} \Big(\prod_{(i,j) \in \vec{P}} g^{a^i_{\mathcal{D}} a^j_{\mathcal{D}}} \Big)$$

Apply it to expression of invariant :

$$\langle I_{\vec{\mathcal{B}}}(T) \rangle_0 = \sum_{\{a_{\mathcal{D}}\}} \Big\langle \prod_{(i,j) \in \vec{P}_{ref\,,\mathbf{2}k}} T^{a^i_{\mathcal{D}}}\, T^{a^j_{\mathcal{D}}} \Big\rangle_0 \prod_{c \in \mathcal{D}} \Big(\epsilon(\vec{P}_{ref\,,2k}, \vec{E}^c(\vec{\mathcal{B}}))^{|c|} \prod_{(i,j) \in \vec{E}^c(\vec{\mathcal{B}})} g^c_{a^i a^j} \Big)$$

$$\langle I_{\vec{\mathcal{B}}}(T)\rangle_0 = \sum_{\{a_\mathcal{D}\}} \sum_{P\in\mathcal{P}_{2k}} \epsilon(\vec{P}_{ref,2k},\vec{P})^{\sum_{c\in\mathcal{D}}|c|} \Big(\prod_{(i,j)\in\vec{P}} g^{a^i_\mathcal{D} a^j_\mathcal{D}} \Big) \times \prod_{c\in\mathcal{D}} \Big(\epsilon(\vec{P}_{ref,2k},\vec{E}^c(\vec{\mathcal{B}}))^{|c|} \prod_{(i,j)\in\vec{E}^c(\vec{\mathcal{B}})} g^c_{a^i_c a^j_c} \Big)$$

Rearrange the terms by colors :

$$\sum_{\{a_\mathcal{D}\}} \sum_{P\in\mathcal{P}_{2k}} \prod_{c\in\mathcal{D}} \Bigg(\epsilon(\vec{P}_{ref,2k},\vec{P})^{|c|} \epsilon(\vec{P}_{ref,2k},\vec{E}^c(\vec{\mathcal{B}}))^{|c|} \Big(\prod_{(i,j)\in\vec{P}} g^{c,\, a^i_c a^j_c} \Big) \Big(\prod_{(k,l)\in\vec{E}^c(\vec{\mathcal{B}})} g^c_{a^k_c a^l_c} \Big) \Bigg)$$

Diagrammatic representation (in terms of oriented $D+1$-colored graphs $\vec{\mathcal{G}}$) :

- Vertices given by D-colored graphs $\vec{\mathcal{B}}$
- Edges of color 0 given by $\vec{P}$

$$\epsilon(\vec{P}_{ref,2k},\vec{P})^{|c|} \epsilon(\vec{P}_{ref,2k},\vec{E}^c(\vec{\mathcal{B}}))^{|c|} \to (-1)^{|c| F_{c/\mathbf{0},\mathbf{even}}(\vec{\mathcal{G}})}$$

$$\sum_{\{a_\mathcal{D}\}} \Big(\prod_{(i,j)\in\vec{P}} g^{c,\, a^i_c a^j_c} \Big) \Big(\prod_{(k,l)\in\vec{E}^c(\vec{\mathcal{B}})} g^c_{a^k_c a^l_c} \Big) \to (-1)^{|c| F_{c/\mathbf{0},odd}(\vec{\mathcal{G}})} N_c^{F_{c/\mathbf{0}}(\vec{\mathcal{G}})}$$

Invariance of the partition function

We find finally

$$\langle I_{\vec{\mathcal{B}}}(T)\rangle_0 = \sum_{\substack{\mathcal{G},\, \mathcal{B}\subset\mathcal{G} \\ |V(\mathcal{G})|=2k}} \prod_{c\in\mathcal{D}} \left((-1)^{|c|} N_c\right)^{F_{c/\mathbf{0}}(\mathcal{G})}$$

Inserting it in the partition function :

$$Z(\{\lambda_{\mathcal{B}}\}) = \sum_{\substack{\mathcal{G} \\ |V(\mathcal{B})|\neq 2\ \forall \mathcal{B}\subset\mathcal{G}}} \frac{1}{n_b(\mathcal{G})!} \Big(\prod_{\mathcal{B}\subset\mathcal{G}} \frac{\lambda_{\mathcal{B}}}{|V(\mathcal{B})|}\Big) \Big(\prod_{c\in\mathcal{D}} \left((-1)^{|c|} N_c\right)^{F_{c/\mathbf{0}}(\mathcal{G})}\Big) \,,$$

The amplitude of each graph is invariant under

$$|c| \to |c| + 1 \text{ and } N_c \to -N_c$$

.

Reminder : The change $|c| \to |c| + 1$ is equivalent to $O(N_c) \to Sp(N_c)$.

The duality holds graphs by graphs in perturbation theory

Invariance of the correlation functions

The expectation value of trace invariants are computed as derivatives of the logarithm of the partition function :

$$|V(\mathcal{B})| > 2 : \quad \langle I_{\vec{\mathcal{B}}}(T)\rangle = -|V(\mathcal{B})| \, \frac{\partial}{\partial \lambda_{\mathcal{B}}} \ln Z(\{\lambda\})$$

$$\langle I_{\vec{\mathcal{B}}}(T)\rangle = \sum_{\substack{\mathcal{G} \text{ connected}, \\ \mathcal{B}\subset\mathcal{G} \text{ marked}, \\ |V(\mathcal{B}')|\neq 2 \ \forall \mathcal{B}'\subset\mathcal{G}}} \frac{1}{n_b(\mathcal{G})!} \Big(\prod_{\substack{\mathcal{B}'\subset\mathcal{G} \\ \mathcal{B}'\neq\mathcal{B}}} \frac{\lambda_{\mathcal{B}'}}{|V(\mathcal{B}')|} \Big) \Big(\prod_{c\in\mathcal{D}} \big((-1)^{|c|} N_c\big)^{F_{c/\mathbf{0}}(\mathcal{G})} \Big) \, .$$

For $g^{\otimes D}(T,T)$, use the Schwinger-Dyson-Equation :

$$\begin{aligned} 0 &= \frac{(-1)^{\sum_c |c|}}{Z} \int [dT] \sum_{a_{\mathcal{D}}} \frac{\partial}{\partial T^{a_{\mathcal{D}}}} \Big(T^{a_{\mathcal{D}}} e^{-S[T]} \Big) \\ &= \Big(\prod_{c\in\mathcal{D}} (-1)^{|c|} N_c \Big) - \langle g^{\otimes D}(T,T)\rangle - \sum_{\substack{\mathcal{B} \text{ conn.} \\ |V(\mathcal{B})|>2}} \lambda_{\mathcal{B}} \langle I_{\vec{\mathcal{B}}}(T)\rangle \, , \end{aligned}$$

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Irreducible representation of O(N) and Sp(N)

We will consider now tensor models invariant under irreducible representations of $O(N)$ or $Sp(N)$.

Tensor product of fundamental representation of $O(N)$ and $Sp(N)$ decomposes into irreducible representations (see Fulton W and Harris J 2004)

Two operations commute with $O(N)$ and $Sp(N)$ action :

- Trace operation $T^{abc} \to T^{abc} g_{ab}$
- Permutation of indices $T^{abc} \to T^{bca}$

To obtain irreducible representation of the groups, we need to reduce these two operations.

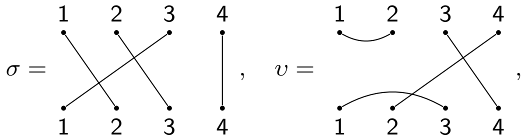
To do so we use the **Brauer Algebra** (Brauer R 1937)

The Brauer algebra $B_D(z)$

The permutation diagrams (in $\mathfrak{S}_D$) are a subset of the Brauer diagrams.

Brauer Diagrams :

- Draw two horizontal rows of vertices labelled $1, 2, \ldots, D$.
- Brauer Diagram given by pairing of the $2D$ vertices.

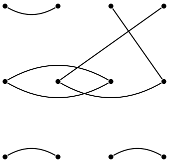
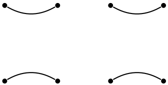


Use the elements of Brauer algebra to :

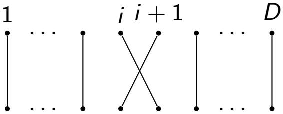
- Symmetrize over irreducible representations of $\mathfrak{S}_D$ (each indexed by a Young tableaux λ).
- Remove the traces of the representations.

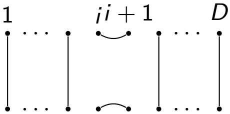
Product of diagrams, product $\beta\upsilon$ of two diagrams obtained by :

- placing β below υ and "straightening" the lines
- deleting loops that appear in the product and multiply by a factor z per loop

$\beta\upsilon =$  $= z$  .

Generators of the algebra :

$\sigma_i =$ 

, $\beta_i =$ 

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Action on Tensor components

V a real N-dimensional vector space with non-degenerate bilinear form $g^{\mathfrak{b}}$ (δ for $\mathfrak{b} = 0$ and ω for $\mathfrak{b} = 1$).

Consider $z = (-1)^{\mathfrak{b}} N$, $N \in \mathbb{N}$

Element $\beta \in B_D((-1)^{\mathfrak{b}} N)$, acts on tensor components $T^{a_1 a_2 \dots a_D}$:

- Place indices $a_1 a_2 \dots a_D$ in the bottom row of β.
- Permute them according to the lines of β
- Contract them with $g^{\mathfrak{b}}$ if they are connected by an arc in the top row.
- Add a factor $g_{\mathfrak{b}}^{a_i a_j}$ for each arc in the bottom row.
- Multiply the result by $(\epsilon(\beta))^{\mathfrak{b}}$, where $\epsilon(\beta)$ is the sign of the pairing induced by β w.r.t the reference pairing $\{(1, D), \dots (2, 2D)\}$

To each element β we define a map

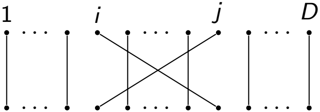
$$(\beta)^{a_1 a_2 \dots a_D}_{b_1 b_2 \dots b_D} = \epsilon(\beta)^{\mathfrak{b}} \prod_{\substack{(i,j) \\ i \text{ in bottom row} \\ \text{connected to } j \text{ in top row}}} \delta^{a_i}_{b_j} \prod_{\substack{(k,l) \\ k \text{ connected to } l \\ \text{by arc in bottom row}}} g_{\mathfrak{b}}^{a_k a_l} \prod_{\substack{(m,p) \\ m \text{ connected to } p \\ \text{by arc in top row}}} g^{\mathfrak{b}}_{b_m b_p}$$

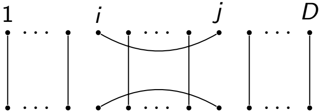
Example of action of β

$$\sigma_{ij} \cdot T^{a_1 \dots a_i \dots a_j \dots a_D} = T^{a_1 \dots a_j \dots a_i \dots a_D} \ ,$$

$$\beta_{ij} \cdot T^{a_1 \dots a_i \dots a_j \dots a_D} = g_{\mathfrak{b}}^{a_i a_j} \ g^{\mathfrak{b}}_{b_i b_j} \ T^{a_1 \dots b_i \dots b_j \dots a_D} \ ,$$

$$\upsilon \cdot T^{a_1 a_2 a_3 a_4} = g_{\mathfrak{b}}^{a_1 a_3} \ g^{\mathfrak{b}}_{b_1 b_2} \ T^{b_1 b_2 a_4 a_2} \ .$$

$\sigma_{ij} =$ 

, $\beta_{ij} =$ 

One can also raise the indices of $(\beta)^{a_1 a_2 \dots a_D}_{b_1 b_2 \dots b_D}$ with g^{ab}

$$(\beta)^{a_1 a_2 \dots a_D, a_{D+1} \dots a_{2D}} = \epsilon(\vec{\beta}, \vec{M}_{ref})^{\mathfrak{b}} \prod_{(i,j) \in \vec{\beta}} g_{\mathfrak{b}}^{a_i a_j} \ .$$

Young projectors

Each irreducible representation is labeled by a Young tableau λ

$$
\begin{array}{|c|c|c|}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
7 & 8 \\
\cline{1-2}
9 \\
\cline{1-1}
10 \\
\cline{1-1}
\end{array} .
$$

Each box represent an index of $T^{a_1 \cdots a_D}$.
Rows represent symmetry properties of T.
Columns represent antisymmetry properties of T.

Define the projector $P_\lambda^{\mathfrak{b}} = a_\lambda . b_\lambda$

$$
(a_\lambda)^{a_1 \cdots a_D}_{b_1 \cdots b_D} = \sum_{\sigma \in A_\lambda} \operatorname{sgn}(\sigma)^{\mathfrak{b}} \prod_{\substack{(i,j) \\ j=\sigma(i)}} \delta^{a_i}_{b_j} \, , \quad (b_\lambda)^{a_1 \cdots a_D}_{b_1 \cdots b_D} = \sum_{\tau \in Q_\lambda} \operatorname{sgn}(\tau)^{b+1} \prod_{\substack{(i,j) \\ j=\tau(i)}} \delta^{a_i}_{b_j} \, .
$$

$\mathfrak{b} = 0$, $P_\lambda^{\mathfrak{b}}$ acts as Young symmetrizer over λ
$\mathfrak{b} = 1$, $P_\lambda^{\mathfrak{b}}$ acts as Young symmetrizer over **transposed** of λ

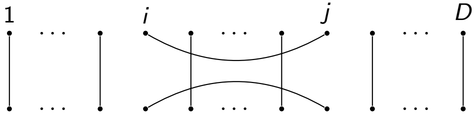
Traceless Projector

Build a traceless projector $\mathfrak{P}_D$ from Brauer algebra s.t (Bulgakov et. al 2022) :

- Commutes with $P_\lambda^\mathfrak{b}$
- Action of A_D on $V^{\otimes D}$ is diagonalizable.
- Kernel $\ker A_D \subset V^{\otimes D}$ is exactly the space of traceless tensors.
- Non-zero eigenvalues are in $(-1)^\mathfrak{b}\mathbb{N}$.

$$\mathfrak{P}_D = \sum_{\alpha \text{ non-zero eigenvalue of } A_D} \left(1 - \frac{1}{\alpha} A_D\right) , \quad A_D = \sum_{1 \leq i < j \leq D} \beta_{ij} \ \in B_D((-1)^\mathfrak{b} N)$$

$\beta_{ij} =$



.

- Sign of directed pairings
- Graded tensor model :
  - The model, colored graphs and invariants
  - Invariance of the model
- Symmetric graded model :
  - Brauer algebra and the traceless projector
  - The model, stranded graphs and invariants
  - Invariance of the model
- Conclusion

The symmetric graded model

To build the model, we consider a tensor $T^{a_1 \cdots a_D}$ with no symmetry properties under the indices and transforming under

$$\boldsymbol{O}(N) \otimes \boldsymbol{O}(N) \otimes \cdots \otimes \boldsymbol{O}(N), \quad \boldsymbol{O}(N) = \begin{cases} O(N), & \mathfrak{b} = 0 \\ Sp(N), & \mathfrak{b} = 1 \end{cases}$$

Only let modes obeying symmetry propagate (of chosen irreducible representation R) :

$$Z = \left[e^{\partial_T (\boldsymbol{C}) \partial_T} e^{-S_{int}(T)} \right]_{T=0}, \quad \langle f(T) \rangle = \frac{\left[e^{\partial_T (\boldsymbol{C}) \partial_T} e^{-S_{int}(T)} f(T) \right]_{T=0}}{\left[e^{\partial_T (\boldsymbol{C}) \partial_T} e^{-S_{int}(T)} \right]_{T=0}}$$

With $\partial_T (\boldsymbol{C}) \partial_T = \frac{\partial}{\partial T^{a_{\mathcal{D}}}} C^{a_{\mathcal{D}} b_{\mathcal{D}}} \frac{\partial}{\partial T^{b_{\mathcal{D}}}}$ and $C = P^{\mathfrak{b}}_{\lambda} . \mathfrak{P}_D$ the projector onto R :

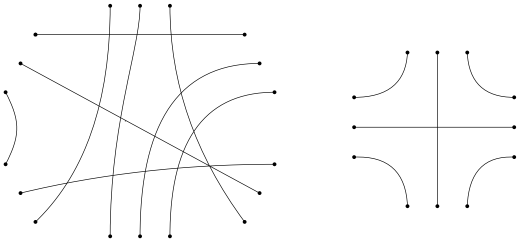
$$C^{a_{\mathcal{D}} b_{\mathcal{D}}} = \sum_{M \in \mathrm{M}\{a_{\mathcal{D}} b_{\mathcal{D}}\}} \gamma_M \, \epsilon(\vec{M}, \vec{M}_{ref,C})^{\mathfrak{b}} \prod_{(i,j) \in \vec{M}} g_{\mathfrak{b}}^{ij} \,,$$

Inserting projector in propagator $\leftrightarrow$ Considering tensor with symmetry of indices

Invariants and Stranded graphs

Invariants of this model cannot be represented in terms of $D-$colored graphs (indices do not have colors anymore).
However they can be represented by stranded graphs $\vec{\mathcal{S}}$ where :

- Each tensor is represented by a set of D nodes given by its indices.
- The set of strands is given by the directed pairing $E(\vec{\mathcal{S}})$ (on all the indices of the tensors in the invariant).



$$I_{\vec{\mathcal{S}}}(T) = \Big(\prod_{(i,j)\in \vec{M}_{ref}} T^{a^i_{\mathcal{D}}} T^{a^j_{\mathcal{D}}} \Big) \, \epsilon(\vec{M}^D_{ref}, \vec{E}(\vec{\mathcal{S}}))^{\mathfrak{b}} \prod_{(k,l)\in \vec{E}(\vec{\mathcal{S}})} g^{\mathfrak{b}}_{kl} \ .$$

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Invariance of the model

The proof is more technical than the first model :

- Write Wick's theorem of the model :

$$\langle T^{a^1_{\mathcal{D}}} \dots T^{a^{2p}_{\mathcal{D}}} \rangle_0 = \sum_{M_0 \in \mathrm{M}_{2p}} \epsilon(\vec{M}_{ref}, \vec{M}_0)^{\mathfrak{b}D} \Big(\prod_{(i,j) \in \vec{M}_0} C^{a^i_{\mathcal{D}} a^j_{\mathcal{D}}} \Big) .$$

- Compute free expectation value of invariants as sum over 2−colored stranded graphs :

$$\langle I_{\mathcal{S}} \rangle_0 = \sum_{\substack{\mathcal{G},\, \mathcal{S} \subset \mathcal{G} \\ |V(\mathcal{G})| = 2pD}} \gamma_{\mathcal{G}} \left((-1)^{\mathfrak{b}} N \right)^{F(\mathcal{G})} .$$

- Expand the partition function and insert $\langle I_{\mathcal{S}} \rangle_0$:

$$Z = \sum_{\substack{\tilde{\mathcal{G}} \\ |V(\mathcal{S})|/D \neq 2 \\ \forall \mathcal{S} \subset \mathcal{G}}} \frac{1}{n_b(\mathcal{G})!} \Big(\prod_{\mathcal{S} \subset \mathcal{G}} \frac{\lambda_{\mathcal{S}}}{|V(\mathcal{S})|/D} \Big) \left((-1)^{\mathfrak{b}} N \right)^{F(\mathcal{G})}$$

Model is thus invariant under the change

$$\mathfrak{b} \to \mathfrak{b} + 1 \text{ and } N \to -N$$

.

Reminder : C is the projector onto an irreducible representations of $O(N)$ or $Sp(N)$:

- labeled by a Young tableau λ for $\mathfrak{b} = 0$
- labeled by transposed of λ for $\mathfrak{b} = 1$

Transpozing λ is equivalent to switching symmetry properties with antisymmetry properties.

The duality links tensor models following an irreducible representation of $O(N)$ to a model with the dual representation of $Sp(N)$ given by flipping the symmetries of the indices of the tensors.

Conclusion

Proven a duality for two types of models :

- Tensor with symmetries : duality obtained by changing symmetry of one color and taking N_c to $-N_c$.
- Tensor models with symmetries : duality obtained by changing total symmetry, taking N_c to $-N_c$ and swapping symmetry of tensors with antisymmetry.

Only looked at the 0 dimensional case.

One could investigate implications of duality in TFTs.

Thank you for your attention !