The small-N expansion: a constructive approach to transseries

based on:

The small-N series in the zero-dimensional $O(N)$ model: constructive expansions and transseries (arxiv 2210.14776)

[D. Benedetti, R. Gurau, H. Keppler, D. Lettera]

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UNIVERSITÄT HEIDELBERG
ZUKUNFT SEIT 1386

The model

What do we do?

We apply techniques from **constructive field theory (LVE)** on the $O(N)$ model in $d = 0$.

$$Z(g,N) = \int_{-\infty}^{+\infty} \left[\prod_{i=1}^{N} d\phi_i\right] e^{-\frac{1}{2}\phi_i\phi_i - \frac{g}{4!}(\phi_i\phi_i)^2} = \int_{-\infty}^{+\infty} [d\sigma]\, e^{-\frac{1}{2}\sigma^2} \frac{1}{\left(1 - \imath\sqrt{\frac{g}{3}}\sigma\right)^{N/2}}$$

- Asymptotic series, transseries, Borel resummation
- The partition function of the $O(N)$ model and Stokes Phenomenon
- Constructive techniques: BKAR formula and Loop Vertex Expansion (LVE)
- The free energy of the $O(N)$

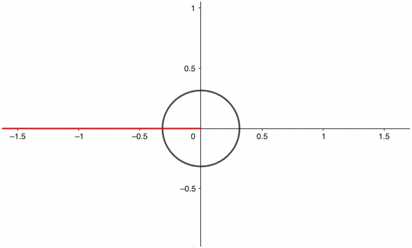
Asymptotic series

Asymptotic series

An asymptotic series is a formal Taylor expansion, in physics often factorially divergent: $A(g) = \sum_{k=0}^{\infty} a_k g^k, \qquad a_k \sim k!$

The series is divergent because of a **bad expansion point**

$$Z(g) = \int_x e^{-\frac{x^2}{2} - gx^4} = \sum_{k=0}^{\infty} \frac{(-g)^k}{k!} \int_x e^{-\frac{x^2}{2}} x^{4k}$$



Borel summation (I)

We make sense of asymptotic series with the theory of Borel resummation

$$A(z) = \sum_{k=0}^{\infty} a_k z^k \qquad B(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k$$

$B(t)$ is the Borel transform of $A(z)$ and has typically a finite radius of convergence! Then the Borel sum of $A(z)$ is

$$f(z) = \frac{1}{z} \int_0^{\infty} dt\ e^{-t/z}\ B(t)\ .$$

We need to address the following question

Let us start with a function $f(z)$ which can be formally expanded in an asymptotic series $A(z)$. Is it always true that the Borel sum of $A(z)$, $\frac{1}{z}\int_0^{\infty} dt\ e^{-t/z}\ B(t)$, is equal to the function $f(z)$ we started with?

Borel summation (II)

The answer to the question is: in general NO!
A typical example is $e^{-1/z}$:

$$A(z) = 0, \quad \to \quad B(t) = 0, \quad \to \quad f(z) = 0 \neq e^{-1/z}.$$

Borel summable function $f(z)$ (along real line)

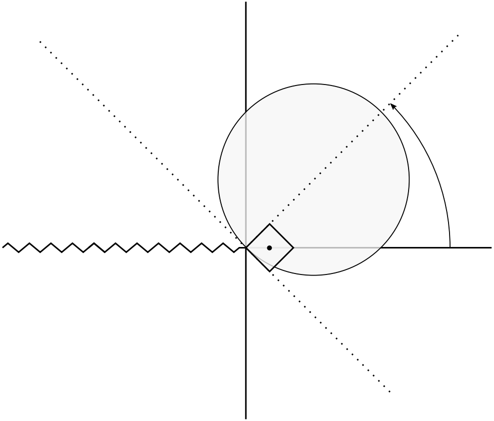
if it is analytic in a disk $\mathrm{Disk}_R = \{z \in \mathbb{C} \mid \mathrm{Re}(1/z) > 1/R\}$ and has an asymptotic series: $f(z) = \sum_{k=0}^{q-1} a_k\, z^k + R_q(z)$ with

$$|R_q(z)| \leq K\, q!\, q^{\beta}\, \rho^{-q}\, |z|^q\,, \qquad z \in \mathrm{Disk}_R\,,$$

Nevanlinna-Sokal theorem guarantees that the Borel sum of $\sum a_k z^k$ is equal to $f(z)$ in Disk_R

Borel summability (III)

The notion of Borel summability is directional



Transseries

A transseries is an object of the following form:

$$F(g) \simeq \sum_{n\geq 0} a_n\, g^n + e^{\frac{c}{g}} g^{\gamma} \sum_{n\geq 0} b_n\, g^n + \dots\, .$$

and clearly it capture also non perturbative physics. Two approaches:

- Via the Écalle's theory of Resurgence when we have a differential equation

$$N(N+2)Z(g,N) + ((8N+24)g+24)\, Z'(g,N) + 16g^2 Z''(g,N) = 0\, .$$

- Via Lefschetz thimbles when we have an integral representation

$$Z(g,N) = \int_{-\infty}^{+\infty} \left(\prod_{a=1}^{N} \frac{d\phi_a}{\sqrt{2\pi}}\right) e^{-S[\phi]}$$

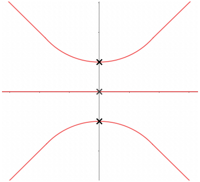
Lefschetz thimble

Any functions $I(g)$ with a contour $\mathcal{C}$ integral representations can be decomposed as

$$I = \int_{\mathcal{C}} dx\ e^{f(x)} a(x) = \sum_i \int_{\mathcal{J}_i} dx\ e^{f(x)} a(x)\ .$$

where $\mathcal{J}_i$ are well chosen **contours**, known as Lefschetz thimbles. They have $\mathrm{Im} f(x)$ =const. and cross critical points $f'(x^\star) = 0$.

$$Z(g) = \int_{-\infty}^{+\infty} \left(\frac{d\phi}{\sqrt{2\pi}}\right)\ e^{-S[\phi]}\ , \qquad S[\phi] = \frac{1}{2}\phi^2 + \frac{g}{4!}\phi^4\ .$$

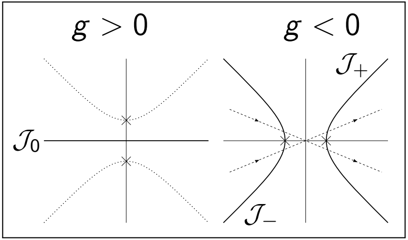


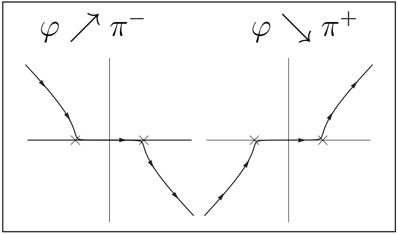
In $Z(g)$, the Thimbles depends parametrically on $g = |g|e^{\imath\varphi}$

Stokes phenomenon (I)

Stokes lines

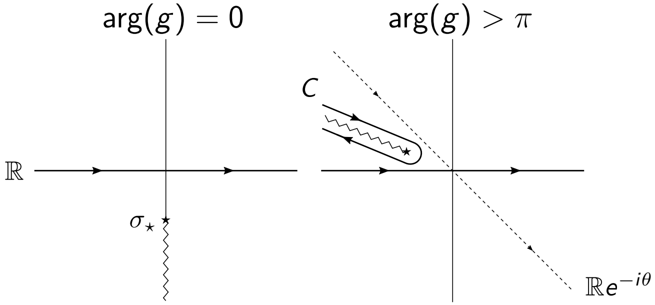
For some choices of φ thimbles cross each others, those are known as Stokes lines. When crossing Stokes lines Thimbles can change discontinuously.





Stokes phenomenon (II)

$$Z(g,N) = \int_{-\infty}^{+\infty} [d\sigma]\, e^{-\frac{1}{2}\sigma^2} \frac{1}{\left(1 - \imath\sqrt{\frac{g}{3}}\sigma\right)^{N/2}}$$



The branch cut singularity approaches the real line when $\arg(g) \to \pi$. When this happens one has to **deform the contour** of integration and **avoid the cut**.

Stokes phenomenon (III)

- $Z^{\mathbb{R}}(g,N)$ integrated along $\mathbb{R}$
- $Z^{C}{}_{\pm}(g,N)$ integrated along the Hankel contour C

Starting at $g>0$

$$Z(g,N)=Z^{\mathbb{R}}(g,N)$$

When we start tilting g in the complex plain $g=|g|e^{\imath\varphi}$

- $\varphi<\pi \quad \rightarrow \quad Z(g,N)=Z^{\mathbb{R}}(g,N)$
- $\varphi>\pi \quad \rightarrow \quad Z(g,N)=Z^{\mathbb{R}}(g,N)+Z^{C}_{\pm}(g,N)$

$$Z^{\mathbb{R}}(g,N)\simeq\sum_{n=0}^{\infty}a_n^{(0)}g^n, \qquad Z^{C}(g,N)\simeq e^{\frac{3}{2g}}\sum_{n=0}^{\infty}a_n^{(1)}g^n$$

For g complex $Z(g,N)=\omega\sum_{n=0}^{\infty}a_n^{(0)}g^n+\eta e^{\frac{3}{2g}}\sum_{n=0}^{\infty}a_n^{(1)}g^n$

Some other results for $Z(g, N)$

- $Z(g, N)$ is absolutely convergent and bounded for $g \in \mathbb{C}_\pi$:

$$|Z(g, N)| \leq \left(\cos \frac{\varphi}{2}\right)^{-N/2}$$

- $Z(g, N)$ is Borel summable along all the directions in $\mathbb{C}_\pi$.
- $Z(g, N)$ can be continued on the entire Riemann surface. However, past $\mathbb{R}_-$ it ceases to be Borel summable.
- A second Stokes line is found at $\mathbb{R}_+$ on the second sheet.
- We can study the analytic continuation and the **Stokes phenomenon of $Z(g, N)$ in the whole Riemann surface.**

The small-N expansion

$$Z(g,N)=\sum_{n\ge 0}\frac{1}{n!}\left(-\frac{N}{2}\right)^n Z_n(g)\ ,\quad Z_n(g)=\int_{-\infty}^{+\infty}[d\sigma]e^{-\frac{1}{2}\sigma^2}\left(\ln(1-\imath\sqrt{\frac{g}{3}}\sigma)\right)^n\ .$$

The Stokes phenomenon for $Z_n(g)$ similar to $Z(g,N)$

- $Z_n(g)$ is analytic in $\mathbb{C}_\pi$ and well bounded. The series has infinite radius of convergence in N.
- The $Z_n(g)$ are Borel summable along all the directions in $\mathbb{C}_\pi$.
- For $g\in\mathbb{C}_\pi$, $Z_n(g)$ has the perturbative expansion:

$$Z_n(g)\simeq\sum_{m\ge n/2}\left(-\frac{2g}{3}\right)^m\frac{(2m)!}{2^{2m}m!}\sum_{\substack{m_1,\dots,m_{2m-n+1}\ge 0\\ \sum km_k=2m,\ \sum m_k=n}}\frac{(-1)^n n!}{\prod_k k^{m_k}m_k!}\equiv Z_n^{\mathrm{pert.}}(g)\ .$$

- $Z_n(g)$ can be continued t in $\mathbb{C}_{3\pi/2}$, and the small-N series is still convergent. Again $\mathbb{R}_-$ is a Stokes line.

Constructive techniques

The small N series of $W(g,N)$

$$W(g,N) = \sum_{n\geq 0} \frac{1}{n!}\left(-\frac{N}{2}\right)^n W_n(g)$$

where $W_n(g)$ cumulants of the random variable $\log(1 - \imath\sqrt{\frac{g}{3}}\sigma)$.
By using techniques from **constructive field theory**

- The BKAR formula
- The Loop Vertex Expansion

we get an integral representation of the $W_n(g)$

Results for $W(g,N)$ (I)

The LVE representation of $W(g,N)$ it's well bounded

- The functions $W_n(g), n \geq 2$ are bounded by:

$$|W_n(g)| \leq \frac{(2n-3)!}{(n-1)!} \left| \frac{g}{3(\cos\frac{\varphi}{2})^2} \right|^{n-1} \to \text{analytic in } \mathbb{C}_\pi .$$

- The series

$$W(g,N) = \sum_{n\geq 1} \frac{1}{n!} \left(-\frac{N}{2}\right)^n W_n(g)$$

  is absolutely convergent in the cardioid domain.

- $W_n(g)$ can be analytically continued to a subdomain of the extended Riemann sheet $\mathbb{C}_{3\pi/2}$. Also the analytically continued series

$$W_\theta(g,N) = \sum_{n\geq 1} \frac{1}{n!} \left(-\frac{N}{2}\right)^n W_{n\theta}(g) ,$$

  is convergent in an **'extended cardioid domain'.**

Results for $W(g, N)$ (II)

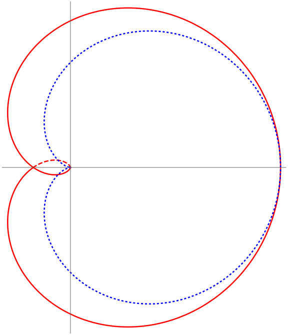


Figure: The dashed blue line is the cardioid domain, the red line is the extended cardioid domain.

$W_n(g)$ and $W(g, N)$ at any fixed complex N are Borel summable along all the directions in the cut complex plane $\mathbb{C}_\pi$.

Asymptotic expansion of $W(g,N)$ (I)

- The LVE is good for bounds and proofs
- Not well suited to get the asymptotic expansion of $W(g,N)$

We use **the Möbius inversion formula:** let π be a partition of the set $\{1,2,\dots n\}$

$$Z_n(g) = \sum_{\pi} \prod_{b\in\pi} W_{|b|}(g) \ , \qquad W_n(g) = \sum_{\pi} \lambda_\pi \prod_{b\in\pi} Z_{|b|}(g) \ ,$$

where $\lambda_\pi = (-1)^{|\pi|-1}(|\pi|-1)!$ Grouping together the partitions with same number of parts n_i of size i

$$W_n(g) = \sum_{k=1}^{n} (-1)^{k-1}(k-1)! \sum_{\substack{n_1,\dots,n_{n-k+1}\ge 0\\ \sum i n_i = n, \sum n_i = k}} \frac{n!}{\prod_i n_i!(i!)^{n_i}} \prod_{i=1}^{n-k+1} Z_i(g)^{n_i}$$

Comments

Transseries expansion of $W_n(g)$:

- In $\mathbb{C}_\pi$ the asymptotic expansion of $Z_i(g)$ is of the perturbative type. Then $W_n(g)$ is just a finite linear combination of Cauchy products of such series.
- Past the Stokes line, each $Z_{i\pm}(g)$ gets an additional contribution from the Hankel contour $Z_{i\pm}(g) = Z_i^{\mathbb{R}}(g) + Z_{i\pm}^{C}(g)$.

A consequence is that

$$W_n(g) \simeq \sum (\ldots) + e^{\frac{3}{2g}} \sum (\ldots) + \ldots + e^{n\frac{3}{2g}} \sum (\ldots)$$

- $W_n(g)$ **has up to** $n-$**instantons contributions in his transseries.**
- $W(g,N)$ **has an infinite tower of instantons**

Conclusions

- New picture of Stokes phenomenon for the of ϕ^4 with intermediate field and Hankel contours
- Constructive techniques good for proofs
- Instantons of $W(g, N)$ past $\mathbb{R}_-$ without formal series
- Disadvantage: more work
- Future: finite dimension $d > 0$ QFT?

Thank you!

Monodromy of $Z(g, N)$

Schematically: $Z(g,N) \sim \omega\, Z^{\mathbb{R}}(g,N) + \eta\, e^{\frac{3}{2g}}\, Z^{\mathbb{R}}(-g,2-N)$, on first sheet $(\omega,\eta) = (1,0)$

$$\begin{aligned}
|\varphi| < \pi : &\quad Z(g,N) = Z^{\mathbb{R}}(g,N)\,, \\
\pi < \varphi < 2\pi : &\quad Z(g,N) = Z^{\mathbb{R}}(g,N) + e^{\frac{3}{2g}}\, Z^{\mathbb{R}}(-g,2-N)\,, \\
2\pi < \varphi < 3\pi : &\quad Z(g,N) = (1+\tilde{\tau})\, Z^{\mathbb{R}}(g,N) + e^{\frac{3}{2g}}\, Z^{\mathbb{R}}(-g,2-N)\,, \\
3\pi < \varphi < 4\pi : &\quad Z(g,N) = \ldots
\end{aligned}$$

We can write a recursion relation for ω and η

$$(\omega_0,\eta_0) = (1,0)\,, \qquad \begin{cases} \omega_{2k+1} & = \omega_{2k} \\ \eta_{2k+1} & = \eta_{2k} + \omega_{2k} \end{cases}\,, \qquad \begin{cases} \omega_{2(k+1)} & = \tilde{\tau}\, \eta_{2k+1} + \omega_{2k+1} \\ \eta_{2(k+1)} & = e^{-\imath\pi(N-1)} \eta_{2k+1} \end{cases}\,.$$

Solved by introducing a transfer matrix:

$$\begin{pmatrix} \omega_{2k} \\ \eta_{2k} \end{pmatrix} = A^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \qquad A = \begin{pmatrix} 1+\tilde{\tau} & \tilde{\tau} \\ e^{-\imath\pi(N-1)} & e^{-\imath\pi(N-1)} \end{pmatrix}\,, \qquad \lambda_A \pm e^{-\imath\pi\frac{N}{2}}$$

The BKAR formula

- Let $i = 1 \dots n$ set of labeled vertices of the complete graph $\mathcal{K}_n$
- The set of edges of $\mathcal{K}_n$ has $n(n-1)/2$ elements.
- $f : [0,1]^{n(n-1)/2} \to \mathbb{R}$ smooth of edge variables x_{ij}

$$f(1,\dots 1) = \sum_{\mathcal{F}} \underbrace{\int_0^1 \cdots \int_0^1}_{|\mathcal{F}| \text{ times}} \left(\prod_{e \in \mathcal{F}} du_e \right) \left[\left(\prod_{e \in \mathcal{F}} \frac{\partial}{\partial x_e} \right) f \right] \left(w_{kl}^{\mathcal{F}}(u_{\mathcal{F}}) \right) ,$$

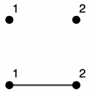
with

$$w_{kl}^{\mathcal{F}}(u_{\mathcal{F}}) = \inf_{e' \in P_{k-l}^{\mathcal{F}}} \{u_{e'}\} > 0 ,$$

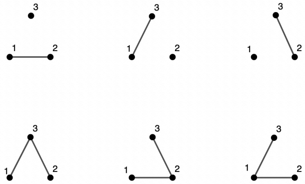
where $P_{k-l}^{\mathcal{F}}$ denotes the unique path in the forest $\mathcal{F}$ joining the vertices k and l, and the infimum is set to zero if such a path does not exist.

Example BKAR formula

Here an example of the BKAR applied on 2 and 3 points:



$$f(1) = f(0) + \int_0^1 du_{12} \frac{\partial}{\partial x_{12}} f(u_{12})$$



$$f(1,1,1) = f(0,0,0)+$$
$$\left(\int_0^1 du_{12} \frac{\partial}{\partial x_{12}} f(u_{12}, 0, 0) + 2 \text{ terms} \right) +$$
$$\left(\int_0^1 du_{13} du_{23} \frac{\partial}{\partial x_{13} \partial x_{23}} f(\inf\{u_{13}, u_{23}\}, u_{13}, u_{23}) + .. \right)$$

Loop Vertex Expansion (I)

Sketch of the derivation of LVE:

$$Z_n(g) = \int [d\sigma]\, e^{-\frac{1}{2}\sigma^2} [\ln(1 - \imath\sqrt{g/3}\,\sigma)]^n \equiv \left[e^{\frac{1}{2}\frac{\delta}{\delta\sigma}\frac{\delta}{\delta\sigma}} V(\sigma)^n \right]_{\sigma=0}$$

We introduce **replicas** and **link paremeters** $x_{ij} = 1$

$$Z_n(g) = \left[e^{\frac{1}{2}\sum_{k,l=1}^{n} \frac{\delta}{\delta\sigma_k}\frac{\delta}{\delta\sigma_l}} \prod_{i=1}^{n} V(\sigma_i) \right]_{\sigma_i=0} = \left[e^{\frac{1}{2}\sum_{k,l=1}^{n} x_{kl}\frac{\delta}{\delta\sigma_k}\frac{\delta}{\delta\sigma_l}} \prod_{i=1}^{n} V(\sigma_i) \right]_{\sigma_i=0, x_{ij}=1} .$$

Then we use BKAR on the x_{ij} link variables

$$Z(g,N) = \sum_{n\geq 0} \frac{\left(-\frac{N}{2}\right)^n}{n!} \sum_{\mathcal{F}\in F_n} \int \prod_{(i,j)\in\mathcal{F}} du_{ij} \left[e^{\frac{1}{2}\sum w_{kl}^{\mathcal{F}} \frac{\delta^2}{\delta\sigma_k\delta\sigma_l}} \left(\prod_{(i,j)\in\mathcal{F}} \frac{\delta}{\delta\sigma_i}\frac{\delta}{\delta\sigma_j} \right) \prod_{i=1}^{n} V(\sigma_i) \right]_{\sigma_i=0}$$

Loop Vertex Expansion (II)

The BKAR is really useful when we want to take logarithms,

$$\sum_{\mathcal{F}} \to \sum_{\mathcal{T}}$$

Also we can take the derivatives explicitly

$$\frac{\delta^d}{\delta\sigma^d} \ln\left(1 - \imath\sqrt{\tfrac{g}{3}}\sigma\right) = (-1)\,\frac{(d-1)!\left(\imath\sqrt{\frac{g}{3}}\right)^d}{\left(1-\imath\sqrt{\frac{g}{3}}\sigma\right)^d}\,,$$

$$W(g,N) = -\frac{N}{2}\left[e^{\frac{1}{2}\frac{\delta}{\delta\sigma}\frac{\delta}{\delta\sigma}} \ln\left(1-\imath\sqrt{\tfrac{g}{3}}\sigma\right)\right]_{\sigma=0}$$

$$-\sum_{n\geq 2}\frac{1}{n!}\left(-\frac{N}{2}\right)^n\left(\frac{g}{3}\right)^{n-1}\sum_{\mathcal{T}\in T_n}\int_0^1\prod_{(i,j)\in\mathcal{T}} du_{ij}\left[e^{\frac{1}{2}\sum w_{ij}^{\mathcal{T}}\frac{\delta^2}{\delta\sigma_i\delta\sigma_j}}\prod_i\frac{(d_i-1)!}{\left(1-\imath\sqrt{\frac{g}{3}}\sigma_i\right)^{d_i}}\right]_{\sigma_i=0}.$$