Matrix Model for Causal Dynamical Triangulations with Ising Model: solution attempts and mathematical limitations

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Motivations and background

- We define a matrix model that describes the Ising Model coupled to the Causal Dynamical Triangulations (CDT). We would like to solve for the partition function in terms of the couplings and study critical behaviours.
- We revisit a problem that appeared on the CDT Matrix Model, which is finding the gaussian average of the character of the square of hermitian matrices.
- It may be useful to study the extension of these notions to tensor models, so we may have a model for dimensions higher than two.

What are our problems?

- There is no known explicit expression for the Clebsch–Gordan and for the Littlewood–Richardson coefficients in terms of the representations involved.
- The use of the character expansion method requires the consideration of representations of $\mathrm{GL}(N)$ which its size grows with N^2.
- To solve for the partition function, we need the determinant of some antisymmetric matrices, but no expression seems to be known.

Main results

- Unitary integral of the product of 4 matrix coefficients in terms of Clebsch-Gordan coefficients.

$$\int_{U(N)} d\Omega\; G^r_{ij}(\Omega)\bar{G}^r_{kl}(\Omega)G^s_{mn}(\Omega)\bar{G}^s_{op}(\Omega) = \sum_{r,m,n,p,q} c^{rm*}_{acp}\, c^{rn}_{\bar{a}\bar{c}p}\, c^{rm}_{bdq}\, c^{rn*}_{\bar{b}\bar{d}q}\, d_r^{-1}$$

- An explicit expression of an $N \times N$ matrix C_m such that $\mathrm{Tr}\, C_m^p = N\delta_{m,p}$ in the large N limit.

$$\lambda_{ts} = e^{\frac{2\pi}{m} i\, s}\mathrm{W}(-e^{\frac{2\pi\, m}{N} i(t-1/2)-1})^{-\frac{1}{m}}\;, \qquad t = 1, ..., N/m \;\text{ and }\; s = 1, ..., m\;.$$

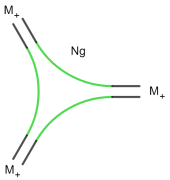
- For a representation R of size n of $\mathrm{GL}(N)$ with $N \gg n$, an expression for the average of character $\langle \chi_R(A^2)\rangle_0$.

$$\langle \chi_R(A^2)\rangle_0 = \frac{\chi_R(\mathbb{1})^2}{\chi_R(C_1)}(1 + O(N^{-2})) = \chi_R(\mathbb{1})(1 + O(N^{-1}))$$

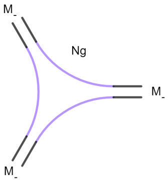
- For finite N, an expression of $\langle \chi_R(A^2)\rangle_0$ in terms of a determinant of an antisymmetric matrix.

$$\langle \chi_{\{h\}}(A^2)\rangle_0 = \frac{1}{Z}\frac{\mathrm{Vol}(U(N))}{(2\pi)^{N-1}} \underset{(i,j)}{\mathrm{Pf}} \frac{(2h_i)!(2h_j)!}{2^{h_i+h_j}} \sum_{\substack{k+l=2h_i\\ u+v=2h_j\\ k+u \text{ is odd}\\ l+v \text{ is odd}}} (-1)^v \frac{(k+u)!!(l+v-2)!!}{k!u!l!v!}$$

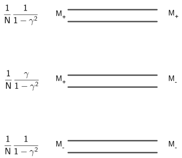
Ising Model on random 2D surfaces via Two Matrix Model
[V.A. Kazakov, 1986]



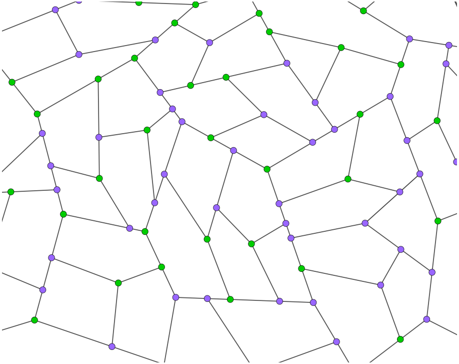
(a) Spin up vertex



(b) Spin down vertex



(c) Interaction edge



(d) Fatgraph shown as a standard graph

Ising Model on random 2D surfaces via Two Matrix Model

- Action for the model with a spin associated to each vertex of the matrix graph.

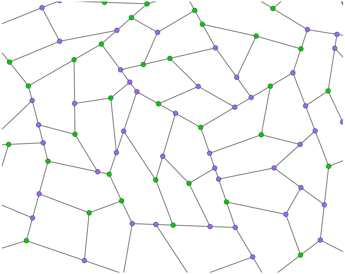
$$S_{IMV} = N \, \mathrm{Tr} \left[\frac{1}{2} M_+^2 + \frac{1}{2} M_-^2 - \gamma M_+ M_- - g M_+^3 - g M_-^3 \right] ,$$

with

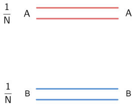
M_+ and M_-: Hermitian matrices,

$\gamma = e^{-2\frac{1}{T}}$, where T is the Ising temperature, and

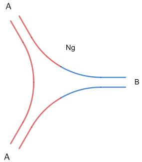
$g = e^{-\Lambda}$, where Λ is a cosmological constant.



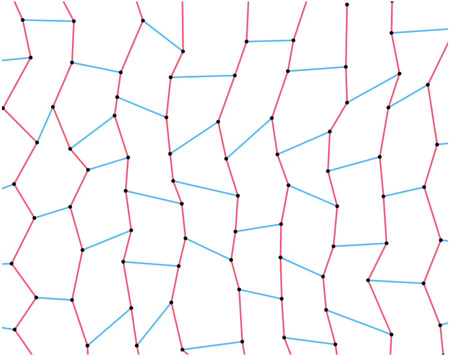
Causal Dynamical Triangulation (CDT) Matrix Model
[D. Benedetti, J. Henson, 2009]



(a) Spacelike edge (red) and timelike edge (blue).



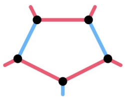
(b) Vertex.



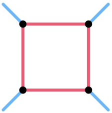
(c) Fatgraph shown as a simple graph.

CDT properties

- The CDT Matrix Model generates graphs with two defining properties: Every vertex has two spacelike edges and one timelike edge; Every face can only have 2 or 0 timelike edges.



(a) A face with 2 timelike edges and 3 spacelike edges.



(b) A face with 0 timelike edges and 4 spacelike edges.

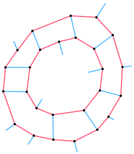
Action for CDT:

$$S_{CDT} = N \,\mathrm{Tr}\left[\frac{1}{2}A^2 + \frac{1}{2}\left(C_2^{-1}B\right)^2 - gA^2B\right] .$$

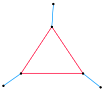
A and B: Hermitian matrices. C_2 is a matrix that satisfies $\mathrm{Tr}\, C_2^p = N\delta_{p,2}$.

CDT properties

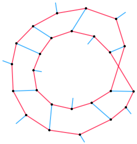
- This leads to the appearance of strips, composed by faces and timelike edges; and borders of strips, composed of vertices and spacelike edges. The strip possibilities are three:



(a) Regular strip



(b) Singular strip



(c) Mobius strip

CDT properties

- All borders have the same number of vertices and spacelike edges.

$$V = E_s$$

- Except for the singular strip, a face with no timelike edge, all strips have the same number of faces and timelike edges.

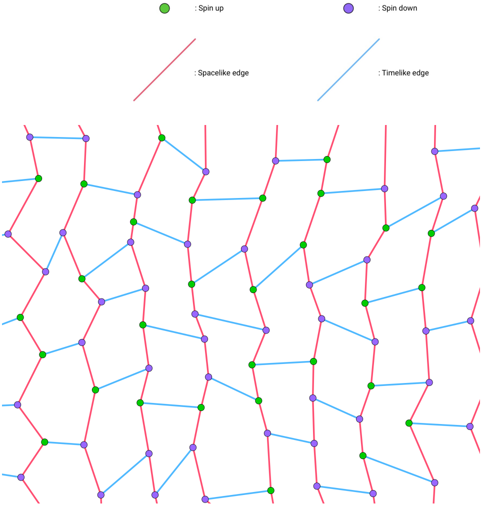
$$F = E_t + F_0$$

- Euler characteristic is simply the number of regular strips:

$$\chi = F - E + V = (F - E_t) + (V - E_s) = F_0$$

- When there are two singular strips connected by two regular strips, we get the sphere topology.
- When there is no singular strips and the regular strips form a loop, we get the torus topology.
- When there is one singular strip and one non-orientable strip or strip border, we get the projective plane topology.
- When there is no singular strip and there are two non-orientable strips or strip borders, we get the Klein bottle topology.

Ising Model with CDT Matrix Model



(a) Green vertices: Spin up. Purple vertices: spin down. Red edges: Spacelike edge. Blue edges: Timelike edge. All vertices have degree 3. Fatgraph shown as a simple graph.

CDT with Ising Model

- Action for CDT coupled with Ising Model over the vertices:

$$S = N\,\mathrm{Tr}\left[\frac{1}{2}A_+^2 + \frac{1}{2}\left(C_2{}^{-1}B_+\right)^2 + \frac{1}{2}A_-^2 + \frac{1}{2}\left(C_2{}^{-1}B_-\right)^2 - \gamma A_+A_- - \gamma(C_2{}^{-1}B_+)(C_2{}^{-1}B_-) - gA_+^2B_+ - gA_-^2B_-\right],$$

where A_+, B_+, A_- and B_- are hermitian matrices and C_2 is the matrix that satisfies $\mathrm{Tr}\,C_2^p = N\delta_{p,2}$.

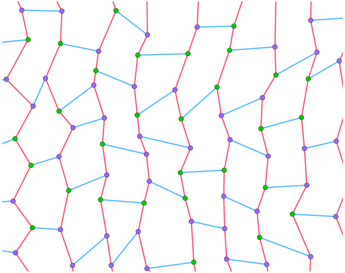


Figure: Fatgraph shown as a standard graph

Brief review of representations

Brief review of representations

- A d-dimensional representation of a group G is roughly a function $R : G \to \mathrm{GL}(d)$ such that for any pair $g_1, g_2 \in G$ it satisfies $R(g_1 g_2) = R(g_1)R(g_2)$.
- The character in R of $g \in G$ is the trace in that representation, $\chi_R(g) = \mathrm{Tr}(R(g))$.
- Representations of $\mathrm{GL}(N)$ can be parameterized by a sequence of N decreasing non negative integers $\{h\} = \{h_1, h_2, ..., h_N\}$, known as shifted weights.
- The size of a representation can be defined by the relation $n = \#h = \sum_i h_i - N(N-1)/2$.

Applying the character expansion method

- The partition function for CDT+Ising Model will be

$$Z = \int dUdV \, e^{-N\mathrm{Tr}[\frac{1}{2}(1-\gamma)U^2+\frac{1}{2}(1+\gamma)V^2-\frac{1}{4}\frac{g^2}{1-\gamma}((U^2+V^2)C_2)^2-\frac{1}{4}\frac{g^2}{1+\gamma}((UV+VU)C_2)^2]} ,$$

where U and V are hermitian matrices, and C_2 defined by $\mathrm{Tr}\, C_2^p = N\delta_{p,2}$ as before.

Up to proportionality constant in N, after applying the character expansion [V. A. Kazakov et al., '95] we get

$$Z \sim \sum_{\{h^1\},\{h^2\}} \left(\frac{Ng^2}{4(1-\gamma)}\right)^{\#h^1/2} \left(\frac{Ng^2}{4(1+\gamma)}\right)^{\#h^2/2} c_{\{h^1\}} c_{\{h^2\}} I_{\{h^1\},\{h^2\}} ,$$

where $U = \Omega_1 \Lambda_1 \Omega_1^\dagger$, $V = \Omega_2 \Lambda_2 \Omega_2^\dagger$, $\Omega_2 = \Omega_1 \Omega$, and

$$I_{\{h^1\},\{h^2\}} = \int d\Lambda_1 d\Lambda_2 \, \Delta(\Lambda_1)^2 \Delta(\Lambda_2)^2 e^{-N\mathrm{Tr}[\frac{1}{2}(1-\gamma)\Lambda_1^2+\frac{1}{2}(1+\gamma)\Lambda_2^2]}$$
$$\int d\Omega_1 d\Omega \, \chi_{\{h^1\}}(\Omega_1(\Lambda_1^2 + \Omega\Lambda_2^2\Omega^\dagger)\Omega_1^\dagger C_2)$$
$$\chi_{\{h^2\}}(\Omega_1(\Lambda_1\Omega\Lambda_2\Omega^\dagger + \Omega\Lambda_2\Omega^\dagger\Lambda_1)\Omega_1^\dagger C_2) .$$

- Given a representation r of $GL(N)$, for Ω in this representation define as $G^r_{ij}(\Omega)$ its matrix coefficients

- One of the formulas that can be used through the character expansion method is the orthogonality

$$\int_{U(N)} d\Omega \; G^r_{ij}(\Omega)\bar{G}^s_{kl}(\Omega) = \delta^{rs}\delta_{ik}\delta_{jl}\frac{1}{d_r}$$

  This formula can be used, for example, in the pure CDT Matrix Model.

- When trying to solve the CDT + Ising Model, one expression that seems necessary is

$$I = \int_{U(N)} d\Omega \; G^r_{ij}(\Omega)\bar{G}^r_{kl}(\Omega)G^s_{mn}(\Omega)\bar{G}^s_{op}(\Omega) \; .$$

Weingarten Calculus

Weingarten Calculus

- The Weingarten Calculus deals with integrals of this type. Recent review in [B. Collins, S. Matsumoto, J. Novak, 2021]
- The mathod basicaly consists of finding the trivial representations that appear in the product of the representations involved.
- The product of two representations α and β can be decomposed into irreducible representations, and there is a transformation V that separates these representations, in terms of Clebsch-Gordan coefficients,

$$V|e_a^{\alpha}\rangle|e_c^{\beta}\rangle = \sum_{r,k,p} c_{acp}^{rk}|e_p^{rk}\rangle \ .$$

With the use of Weingarten Calculus we find that the integral can be expressed as

Integral result

$$I = \sum_{r,m,n,p,q} c_{acp}^{rm*} c_{\bar{a}\bar{c}p}^{rn} c_{bdq}^{rm} c_{\bar{b}\bar{d}q}^{rn*} d_r^{-1} \ .$$

- An algorithm is known but it is not enough, we would like a closed formula.

Revisiting the CDT Matrix Model

CDT partition function and average of A^2

- We want to solve the partition function for the couplings

$$Z = \frac{1}{Z_0} \int dA \; e^{-N\mathrm{Tr}\left[\frac{1}{2}A^2 - \frac{g^2}{2}(A^2 C_2)^2\right]}$$

- In terms of characters

$$Z = \sum_{\{h\}} g^n \frac{\chi_{\{h\}}(C_2)^2}{\chi_{\{h\}}(\mathbb{1})} \langle \chi_{\{h\}}(A^2) \rangle_0$$

- If it was just character of A it would be easier to solve, but the power is trouble.
- Conjecture [D. Benedetti, J. Henson, 2009]

$$\langle \chi_{\{h\}}(A^2) \rangle_0 = k_N \frac{1}{N^{\#}} \prod_{\epsilon=0}^{3} \Delta(2h^{(\epsilon)})^2 \prod_i (2h_i^{(\epsilon)})!!$$

Where the set of integers $\{h\}$ has been divided into four sets $\{h^{(0)}\}, ..., \{h^{(3)}\}$ according to equivalence modulo 4.

Counter example

- Conjecture [D. Benedetti, J. Henson, 2009]

$$\langle \chi_{\{h\}}(A^2) \rangle_0 = k_N \frac{1}{N^\#} \prod_{\epsilon=0}^{3} \Delta(2h^{(\epsilon)})^2 \prod_i (2h_i^{(\epsilon)})!!$$

- For the trivial representation, given by $\{h\} = \{3, 2, 1, 0\}$, the proportionality factor is $k_N = 1/768$:

$$1 = k_N \cdot 1 \cdot \prod_{i=0}^{3} (2i)!! = k_N \cdot 768 \; .$$

- For the defining representation, given by $\{h\} = \{4, 2, 1, 0\}$, the proportionality factor is $k_N = 1/(8 \cdot 768)$:

$$4 = k_N \frac{1}{4} (4-0)^2 \frac{(2 \cdot 4)!!}{(2 \cdot 3)!!} \prod_{i=0}^{3} (2i)!! = k_N \cdot 4 \cdot 8 \cdot 768 \, .$$

A contradiction.

Properties of C_m

- C_m is a matrix satisfying at large N the identity

$$\mathrm{Tr}\, C_m^p = N\delta_{m,p} \tag{1}$$

For each $p = 1, ..., N$, equation (1) is an equation on the eigenvalues of C_m.

- Using the Girard–Newton formulae, we find the characteristic polynomial

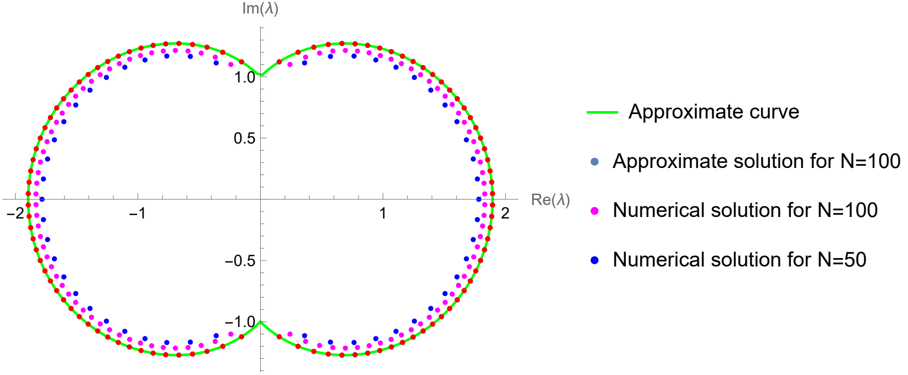
$$f_m(z) = \sum_{r=0}^{N/m} \frac{(-1)^{mr} N^r}{(-m)^r r!} (-\lambda)^{N-mr}$$

Solving for its zeros, we find that for even N, the N eigenvalues are given by

Large N result

$$\lambda_{ts} = e^{\frac{2\pi}{m} i\, s} \mathrm{W}(-e^{\frac{2\pi m}{N} i(t-1/2)-1})^{-\frac{1}{m}}\,, \qquad t = 1, ..., N/m \text{ and } s = 1, ..., m\,.$$

$\mathrm{W}(z)$ is the principal value of a function satisfying $\mathrm{W}^{-1}(z) = ze^z$.
For N odd, the roots are the same as for $N-1$, plus 0 as an additional root.


Im(λ)
Re(λ)
1.0
0.5
−0.5
−1.0
−2
−1
1
2
Approximate curve
Approximate solution for N=100
Numerical solution for N=100
Numerical solution for N=50

Symmetric Group and Schur-Weyl duality for $\langle \chi_R(A) \rangle_0$

Symmetric Group method

- Using Schur-Weyl duality and character orthogonality:

$$\chi_R(A) = \frac{1}{n!}\sum_{\sigma\in S_n}\bar{\chi}_R(\sigma)\mathrm{Tr}(\sigma A^{\otimes n}), \tag{2}$$

  where n is the size of the representation.

- Using Wick contractions:

$$\langle A^{\otimes n}\rangle_0 = N^{-\frac{n}{2}}\sum_{\gamma\in[2^{\frac{n}{2}}]}\gamma .$$

- Using (2) for C_2:

$$\sum_{\gamma\in[2^{\frac{n}{2}}]}\chi_R(\gamma) = n!N^{-\frac{n}{2}}\chi_R(C_2) .$$

- After more use of orthogonality relations and Schur-Weyl duality:

$$\langle\chi_R(A)\rangle_0 = n!\frac{d_R}{l_R}\chi_R(C_2) = \frac{\chi_R(\mathbb{1})\chi_R(C_2)}{\chi_R(C_1)} ,$$

  where $d_R = \chi_R(\mathbb{1})$ and l_R are the dimensions of the $\mathrm{GL}(N)$ and S_n representations, respectively, and $\mathrm{Tr}\,C_m^p = N\delta_{p,m}$.

Symmetric Group and Schur-Weyl duality for $\langle \chi_R(A^2) \rangle_0$

- Defining the partial trace $\mathrm{P}_2\mathrm{Tr}(B)^{j_1\cdots j_n}_{i_1\cdots i_n} = \sum_{k_1\cdots k_n} B^{j_1\cdots j_n k_1\cdots k_n}_{i_1\cdots i_n k_1\cdots k_n}$, which is a contraction over the last n variables, for $N \gg n$ we found that

$$\sum_{\gamma\in[2^n]} \mathrm{P}_2\mathrm{Tr}(\gamma\ \alpha) = \sum_{\rho\in S_n} \mathrm{Tr}(\rho)\ \rho\ (1 + O(N^{-2}))\ .$$

- As an example, consider the permutations $[2^2]$

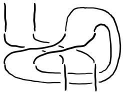
(a) (1 2)(3 4)

(b) (1 3)(1 4)

(c) (1 4)(2 3)

The partial traces associated to these with α are

(a) (1)(2)



(b) $N^2(1)(2)$

(c) $N(1\,2)$

- This way, we check that

$$\sum_{\gamma\in[2^2]} \mathrm{P}_2\mathrm{Tr}(\gamma\ \alpha) = [1 + N^2](1)(2) + N(1\,2) = [N^2(1)(2) + N(1\,2)][1 + O(N^{-2})]$$

Symmetric Group method

- Using Schur-Weyl duality and character orthogonality,

$$\chi_R(A^2) = \frac{1}{n!}\sum_{\sigma\in S_n} \bar{\chi}_R(\sigma)\mathrm{Tr}(\sigma(A^2)^{\otimes n}),$$

we get to a result similar to the previous, but with $\mathrm{P}_2\mathrm{Tr}(\gamma\ \alpha)$ instead of just γ:

$$\langle \chi_R(A^2)\rangle_0 = N^{-n}\frac{d_R}{l_R}\sum_{\gamma\in[2^n]} \chi_R(\mathrm{P}_2\mathrm{Tr}(\gamma\ \alpha))\ .$$

- Using the partial trace result leads to

Large N result

$$\langle \chi_R(A^2)\rangle_0 = \frac{\chi_R(\mathbb{1})^2}{\chi_R(C_1)}(1+O(N^{-2}))$$

- But this result is only for $N \gg n$, while we expect that we need the result for $n \sim N^2$.
- This might be related to Walled Brauer Algebra. If we reflect half of the diagram, it will be a diagram of this algebra, and the partial trace is similar to an operation possible on this algebra.

De Bruijn's Formula with Pfaffians

$$\frac{1}{N!}\int d\mu(X)\mathrm{det}_{i,j}f_i(x_j)\mathrm{Pf}_{i,j}A(x_i,x_j) = \mathop{\mathrm{Pf}}_{(i,j)}\int d\mu(x)d\mu(y)f_i(x)A(x,y)f_j(y)$$

Pfaffian method

- Average by the integral over the diagonal part:

$$\langle \chi_r(A^2) \rangle = \frac{\mathrm{vol}(U(N))}{N!\,(2\pi)^N} \int dX \prod_{i<j}^{N} (x_i - x_j)^2 \frac{\det x_i^{2(m_j+N-j)}}{\prod (x_i^2 - x_j^2)} e^{-\frac{N}{2}\mathrm{Tr} X^2}$$

We can turn the N integrals into a Pffaffian and 2 integrals by de Bruijn's formula [N. G. de Bruijn, '55], with a principal value integral

$$\langle \chi_{\{h\}}(A^2) \rangle_0 = \frac{1}{Z_0} \frac{\mathrm{Vol}(U(N))}{(2\pi)^N} \mathrm{Pf} \int dxdy \; e^{-\frac{N}{2}(x^2+y^2)} x^{2Nh_i} y^{2Nh_j} \frac{x-y}{x+y} \; .$$

- Solving the integral with a damping and by *saddle point approximation* we get

$$\langle \chi_{\{h\}}(A^2) \rangle_0 = \frac{1}{Z_0} \frac{\mathrm{Vol}(U(N))}{(2\pi)^N} \left(\frac{2\pi}{N} \right)^{\frac{N}{2}} e^{-N \sum_i h_i} \prod_i (2h_i)^{Nh_i} 2^{\frac{N}{2}} \mathrm{Pf} \frac{h_i + h_j}{h_i - h_j}$$

- Are we allowed to do this? In which conditions we can apply the saddle point approximation while having poles?
- Divergences and problems.

Pfaffian method

- *Exact* solution with $a = (\alpha + \beta)/\sqrt{2}$ and $b = (-\alpha + \beta)/\sqrt{2}$

$$\langle \chi_{\{h\}}(A^2) \rangle_0 = \frac{1}{Z_0} \frac{\mathrm{Vol}(U(N))}{(2\pi)^N} \mathrm{Pf} \frac{2\pi}{N} \frac{\partial^{2h_i}}{\partial \alpha^{2h_i}} \frac{\partial^{2h_j}}{\partial \beta^{2h_j}} a\, e^{N\frac{a^2}{2}} \int_0^b db'\ e^{N\frac{b'^2}{2}} \Big|_{\alpha,\beta=0} ,$$

which we can evaluate in terms of summations

Finite N result

$$\langle \chi_{\{h\}}(A^2) \rangle_0 = \frac{1}{Z_0} \frac{\mathrm{Vol}(U(N))}{(2\pi)^{N-1}} \underset{(i,j)}{\mathrm{Pf}} \frac{(2h_i)!(2h_j)!}{2^{h_i+h_j}} \sum_{\substack{k+l=2h_i \\ u+v=2h_j \\ k+u \text{ is odd} \\ l+v \text{ is odd}}} (-1)^v \frac{(k+u)!!(l+v-2)!!}{k!u!l!v!}$$

Conclusion

- We were able to express the unitary integral of the product of 4 matrix coefficients in terms of Clebsch-Gordan coefficients, but a general expression of these coefficients is still unknown.
- Got an explicit expression of an $N \times N$ matrix C_m such that $\mathrm{Tr}\, C_m^p = N\delta_{m,p}$ in the large N limit, which should be helpful for applying numerical methods to the model.
- For a representation R of size n of $\mathrm{GL}(N)$ with $N \gg n$, we found an expression for the average of character $\langle \chi_R(A^2) \rangle_0$. We would like to extend this for the case when n also grows.
- For finite N and n, we found an expression of $\langle \chi_R(A^2) \rangle_0$ in terms of a determinant of an antisymmetric matrices. Solving this determinant would likely be the closest to a good solution.

Thank you!