Random Tensor Networks (RTNs) with local Haar-averaging

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Outline

1. Entanglement measures
2. Entanglement and geometry
3. Random Tensor Networks (RTNs)
4. RTNs with "local" Haar-average

Joint w.i.p. with Luca Lionni
1. Entanglement measures

Bipartite. $\mathcal{H}_1 \otimes \mathcal{H}_2 \ni |14\rangle = \sum_{i_1, i_2} M_{i_1 i_2} |i_1\rangle \otimes |i_2\rangle$

"Entanglement" w features of $|14\rangle$ that are not intrinsic to any one of the subsystems 1, 2.

A reasonable def. (among others):

set of Local Unitary (LU) invariants
i.e. $U(D_1) \otimes U(D_2)$ invariants

Singular Value Decomposition: $M = U_1^\dagger D U_2$
Reduced states \( \{ \) 
\[ \begin{align*}
E_1 &= \text{Tr}_2 (|1\rangle \langle 4|) = \mathbb{M} M^+ \\
E_2 &= \text{Tr}_1 (|1\rangle \langle 4|) = M^+ \mathbb{M}
\end{align*} \]
\( \} \)

\( LU \) invariants of \( |14\rangle \rangle \) \( \iff \) singular values of \( \mathbb{M} \)
\( \iff \) \( \text{Spec} (M^+ \mathbb{M}) = \text{Spec}(\mathbb{M} M^+) \)
\( \iff \) \( \{ \text{Tr} ((M^+ \mathbb{M})^m), m \in \mathbb{N}^* \} \)

"Trace invariants"

\( M_{i_1 i_2} \sim \bigwedge \quad \overline{M}_{i_1 i_2} \sim \bigwedge \)

\( \text{Tr}_n (M^+ \mathbb{M}) \)
\( \text{Tr}_n ((M^+ \mathbb{M})^2) \)
\( \text{Tr}_n ((M^+ \mathbb{M})^3) \)
Rényi entropy: \( S_m (e^1) = \frac{1}{1 - m} \ln T_m (e^m) \)

Von Neumann entropy: \( S_{vn} (e^1) = - T_n (e^1 \ln e^1) \)

\[ \ln \left( \text{rank } e^1 \right) \geq S_{vn} \geq S_2 \geq S_\infty = - \ln \frac{1}{\lambda_{\text{max}}} \]

Example: Bell state \( |4\rangle = \frac{1}{\sqrt{D}} \sum_{i=1}^{D} 1i \otimes 1i \)

\( e_1 = e_2 = \frac{1}{D} 11_D \)

Flat entanglement spectrum: \( \text{Spec} (e^1) = \left\{ \frac{1}{D} \right\} \)

\( S_0 = S_1 = S_2 = S_\infty = \ln D \)
**Multipartite entanglement.**

\[ |\psi\rangle = \sum_{i_1, i_2, \ldots, i_q} \text{Tr}_{i_1, i_2, \ldots, i_q} (|i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_q\rangle) \]

"Entanglement spectrum"

\[ \{ LU \text{ invariants} \} \leftrightarrow \{ \text{Tr}_B (\bar{T}, T), B \text{ q-colored bipartite diag.} \} \]

\( q = 3 \)

\[ T \sim \begin{array}{c}
\otimes \\
\end{array} \]

\[ \bar{T} \sim \begin{array}{c}
\otimes \\
\end{array} \]

\( n = 1 \)

\[ \begin{array}{c}
\otimes \\
\end{array} \]

\( n = 2 \)

\[ \begin{array}{c}
\otimes \\
\end{array}, \begin{array}{c}
\otimes \\
\end{array}, \begin{array}{c}
\otimes \\
\end{array} \]

\[ \begin{array}{c}
\otimes \\
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\end{array}, \begin{array}{c}
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\end{array}, \begin{array}{c}
\otimes \\
\end{array} \]

Factorial growth of \( \# \{ \text{invariants} \} \) with \( n \).

[Ben Geloun, Ramgoolam]
Parametrization in terms of permutations.

\[ \mathbf{Z} = (id, id, id) \in S_2^\times 3 \]

B with 2n modes and 9 colors

\[ \mathbf{Z} = (z_1, ..., z_9) \in S_2^\times 9 \]

\[ \text{up to } z_i \rightarrow \mu z_i \, \forall \]

\[ \text{Tr}_B (\mathbf{\bar{T}}, \mathbf{T}) = \text{Tr}_{\mathbf{Z}} (\mathbf{\bar{T}}, \mathbf{T}) \]

**Q** Which of those many invariants are most relevant to characterize the multipartite entanglement structure of many-body quantum systems?
Entanglement and geometry.

Many-body physics: ground states of gapped local Hamiltonians tend to be highly atypical.

\[ S_{VN}(\mathcal{C}_A) \sim \text{const} \times |\mathcal{A}| \]

Area law

Tensor Networks with a polynomial \# of parameters in system size, which obey area laws by construction.
Relativistic QFT.

Unruh effect / Bisognano–Wichman theorem.

\[ e_A = \frac{e^{-\beta \mathcal{H}_A}}{z} \]

with \( \beta = 2\pi \) boost Hamiltonian \( \mathcal{H}_A \).

Universal divergence in \( S_{\mathcal{V}_0}(e_A) \lesssim \text{Area}(\partial A) \).

Heuristic derivations of GR \[ [\text{Jacobson '95, '15}] \]

\[ \delta S_S(e_A) = \gamma \delta \text{Area}(\partial A) \]

and universal, finite \( \Rightarrow \) Einstein's eqs.

"entanglement equilibrium hypothesis"
Holographic version \((\text{AdS/CFT})\).

\[
S_{\text{wN}}(A) = \frac{\text{Area}(\Sigma_A)}{4G}
\]

\(\Sigma_A\): extremal surface.

RT formula \(\Rightarrow\) Einstein’s eqs. up to 2nd order

[ Faulkner et al. 2017]
Conclusion. Bipartite entanglement and area laws play a central role in many-body physics, relativistic QFT and QG.

Question. Does this interplay between geometry and entanglement extends to the multipartite setting?

Investigate this question on simple toy-models: Random Tensor Networks
RTNs \cite{Hayden:2016}

Graph $G = (V,E)$ with:

$V = V_{\text{bulk}} \cup V_{\partial}$

$\{ \bullet \}$

$\{ \circ \}$

Associate Hilbert space $\mathcal{H}_{e,x}$ to each half-edge $(e,x)$.

"Bond dimension" of network.

$$\dim \mathcal{H}_{e,x} = D$$

Edge data: $|\Phi\rangle = \bigotimes_{e=(x,y), e \in E} \frac{1}{\sqrt{D}} \sum_{i=1}^{D} |i\rangle_x \otimes |i\rangle_y$

Vertex data: $|\Psi\rangle = \bigotimes_{x \in V_0} |\Psi\rangle_x$ with $|\Psi\rangle_x \in \otimes_{e \text{ incident to } x} \mathcal{H}_{e,x}$
Tensor Network state:

\[ |TN\> = (\mathcal{D}_v \otimes \langle 41 |) |\Phi\> \]

Leavers on the boundary \( V_2 \)

Random TN: take \( \{ |4\>_x \} \) to be independent, Haar-distributed random vectors i.e.

\[
\mathbb{E} \left[ 14 | 4 \>_x \right] \otimes^m = \int dU \left[ U^+ 10 \otimes 01 U \right] \otimes^m
\]

\[
= \frac{1}{D(D+1) \cdots (D+m-1)} \sum_{\sigma \in S_m} R_{x \cdot} (\sigma) \leftarrow \text{Permutation operator acting on } D^\otimes
\]
Trace observables \rightarrow classical spin models

\[ \mathcal{G}(A_k, p_k) := \sum_{\sigma \in S_m} E(\sigma) \]

where \[ E(\sigma) = \sum_{e = (x, y) \in E} d(\sigma(x), \sigma(y)) \]

\[ -\ln D \times E(\sigma) \]

Cayley distance on \( S_m \)

[Haydn et al., '16]
Generalized spin model at inverse temp. $\beta = \frac{1}{\ln D}$

$D \to \infty \quad \Rightarrow \quad \beta \to 0$

Hence: at leading order, $O(\theta \kappa_1, \rho \kappa_1)$ is determined by ground states of the model.

**Ex.**

$\mathbb{E} Tr C_A^2$

$(m=2, \ m=2)$

$S_2 = \{ \text{id}, (12) \}$

$\frac{-\ln D \times E_{J\omega\chi}}{\text{det} g_{\sigma}}$

$\mathbb{E} tr C_A^2 = \sum_{\sigma: V \to \{+, -\}} e^{-\ln D |\chi_\sigma|}$

$= \delta_{g, i.c.} e^{(1 + O(1/D))}$

$\Rightarrow$ Area law

$\mathbb{E} S_2 (C_A) \approx |\chi_A| \ln D$
Idea. Keep local entanglement structure at each vertex fixed i.e. average over $LU$ instead of full unitary group of $H_{cl}$.

\[ H_x = \bigotimes_{e} H_{e,x} \]

Hopes:
- Identify which entanglement structures are responsible for area laws.
- Obtain richer entanglement spectra than in the standard case (non-flat spectrum).
Seed state on a vertex:

\[ |T\rangle_x = \sum_{i_1, \ldots, i_q} T_{i_1 \ldots i_q} |i_1\rangle \otimes \cdots \otimes |i_q\rangle \]

Fixed tensor

Average over \( U(D)^{\otimes q} \): \( \mathcal{U}(D)^{\otimes q} \):

\[ |\Psi\rangle_x \sim \text{random state in equivalence class of states with same entanglement structure as } |T\rangle_x \]

\[
E[|\Psi\rangle_x \langle \Psi|] := \int dU^{(1)} \cdots dU^{(q)} \left[ \left( \otimes_{c=1}^{q} U^{(c)^{+}} \right) |T\rangle_x \langle T| \left( \otimes_{c=1}^{q} U^{(c)} \right) \right]_{\text{seed}}
\]

LU average
Weingarten calculus \textsuperscript{[Collins et al. '00s]}

\[ \mathbb{E}\left[ |\psi_\times_\times|^{\otimes n} \right] = \sum_{\sigma_1, \ldots, \sigma_n} F_T(\vec{\sigma}) \hat{I}(\vec{\sigma}) \]

\text{state-dep. weight}
\text{pattern of } S\text{-contractions (stranded graph)}

\text{with}
\[ \hat{I}(\vec{\sigma}) = \prod_{c=1}^{9} \delta_{i_c \sigma_c}, i_{c(0)} \]

\[ F_T(\vec{\sigma}) = \sum_{\vec{z} \in S_m} \text{Tr}_{\vec{z}}(\vec{T}, \vec{T}) \prod_{c=1}^{9} \omega^{(0)}(\sigma_c z_c^{-1}) \]

\text{entanglement invariant (bubble)}
\text{Weingarten function}
Asymptotics of Weingarten's function:

\[ D \omega^{(D)}(\sigma z^{-1}) = D^{-d}(\sigma z) \cdot \Pi (\sigma z^{-1}) \left( 1 + O(\frac{1}{D^2}) \right) \]

Cayley distance

Möbius function
(rational number)

For some family of entangled seed states, this yields:

\[ \# \{ \text{connected comp. of } B_2^{\sigma} \} \]

\[ F_T(\sigma) = \sum_{\tilde{z}} D_{\tilde{z}} \]

minimal
when \( \tilde{z}_1 = \tilde{z}_2 = \cdots = \tilde{z}_q \)

\[ \left( \prod_{c=1}^{q} \Pi (\sigma_c z_c^{-1}) \right) \left( \sum_{c=1}^{q} d(\sigma_c, z_c) - \omega_T(\tilde{z}) \right) \]

minimal
when \( \tilde{z} = \tilde{z}_1 = \tilde{z}_2 = \cdots = \tilde{z}_q \)

state-dependent connection
Examples of seed states:

1. GHZ state:
   \[ |T\rangle = D^{(q-1)/2} \sum_{i=1}^{D} |i_1 \otimes \cdots \otimes i_q \rangle \]

   \[ \omega_T(\tilde{\sigma}) = 0 \]

2. "Cyclic" state:
   \[ |T\rangle = D^{9/4} \]

   \[ \omega_T(\tilde{\sigma}) = g_{\frac{9}{4}}(B_{\tilde{\sigma}}) \]

3. "Complete graph" state:
   \[ |T\rangle = D^{9/4} \]

   \[ \omega_T(\tilde{\sigma}) = \frac{1}{q-1} \omega_{\text{Guran}}(B_{\tilde{\sigma}}) \]
Rényi-$m$ entropy $\leftrightarrow$ generalized "spin" model $(S_m)$

- \[ \gamma_1 = \gamma_2 = \ldots = \gamma_9 = \frac{1}{9} \]

- Ising configurations

- Other contributions

- Vertex defects

- L9/21 type

Ex. $GHZ, \ m=2$. Energy functional:

$$E = E_{\text{Ising}} + 2 \gamma_1 + \sum_{\gamma_2} V_0 \text{ with } V_0 = \#\{\text{defects of type } \gamma_2\}$$

$$\Rightarrow \quad E S_2 (P_A) = c_A \ln D \text{ with } c_A \leq 18_A$$

\( \Rightarrow \text{non-flat spectrum.} \)
Conclusion.

- RTNs with reduced amount of averaging can be analyzed thanks to Weingarten calculus.
- Allows tunable local entanglement structure at each vertex of the network.
- Rényi entropy evaluation maps to generalized spin model; permutations assigned to half-edges, with determining interaction potential at a vertex.
- Interesting new features e.g. non-flat entanglement spectra.

... w.i.p ... stay tuned!