Random Tensor Networks (RTNs) with local Haar-averaging

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Outline

① Entanglement measures

② Entanglement and geometry

③ Random Tensor Networks (RTNs)

④ RTNs with "local" Haar-average

Joint w.i.p with Luca Lionni

① Entanglement measures

Bipartite. $\underset{D_1}{\mathcal{H}_1} \otimes \underset{D_2}{\mathcal{H}_2} \ni |\psi\rangle = \sum_{i_1, i_2} M_{i_1 i_2} \; |i_1\rangle \otimes |i_2\rangle$

"Entanglement" ~ features of $|\psi\rangle$ that are not intrinsic to any one of the subsystems 1, 2.

A reasonable def. (among others):

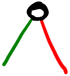
$$\boxed{\begin{array}{l} \text{set of Local Unitary (LU) invariants} \\ \text{i.e. } U(D_1) \otimes U(D_2) \text{ invariants} \end{array}}$$

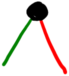
Singular Value Decomposition: $M = U_1 \underset{\text{diagonal}}{D} U_2$

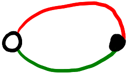
Reduced states $\begin{cases} \rho_1 = \mathrm{Tr}_2(|\psi\rangle\langle\psi|) = MM^\dagger \\ \rho_2 = \mathrm{Tr}_1(|\psi\rangle\langle\psi|) = M^\dagger M \end{cases}$

$$
\begin{aligned}
\{\text{LU invariants of } |\psi\rangle\} &\iff \{\text{singular values of } M\} \\
&\iff \mathrm{Spec}(M^\dagger M) = \mathrm{Spec}(MM^\dagger) \\
&\iff \{\mathrm{Tr}((M^\dagger M)^n),\ n \in \mathbb{N}^*\}
\end{aligned}
$$

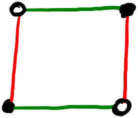
"Trace invariants"

$M_{i_1 i_2} \sim$ 

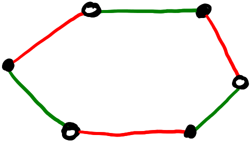
$\overline{M}_{i_1 i_2} \sim$ 



$\mathrm{Tr}(M^\dagger M)$



$\mathrm{Tr}((M^\dagger M)^2)$



$\mathrm{Tr}((M^\dagger M)^3)$

Rényi entropy: $(n \geq 2)$

$$\boxed{S_n(\rho_1) = \frac{1}{1-n} \ln \mathrm{Tr}(\rho_1^n)}$$

Von Neumann entropy:

$$\boxed{S_{vN}(\rho_1) = -\mathrm{Tr}(\rho_1 \ln \rho_1)}$$

$$\underbrace{\ln(\mathrm{rank}\, \rho_1)}_{S_0} \geq \underbrace{S_{vN}}_{S_1} \geq S_2 \geq S_\infty = -\ln \lambda_{max}$$

<u>Example.</u> Bell state $|\psi\rangle = \frac{1}{\sqrt{D}} \sum_{i=1}^{D} |i\rangle \otimes |i\rangle$

$$\rho_1 = \rho_2 = \frac{1}{D} \mathbb{1}_D$$

$\rightsquigarrow$ flat entanglement spectrum: $\mathrm{Spec}(\rho_1) = \{\frac{1}{D}\}$

$$S_0 = S_1 = S_2 = S_\infty = \ln D$$

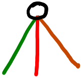
Multipartite entanglement

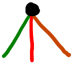
$$|\psi\rangle = \sum_{i_1, i_2, \dots, i_q} T_{i_1, i_2, \dots, i_q} \, |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_q\rangle$$

"Entanglement spectrum"

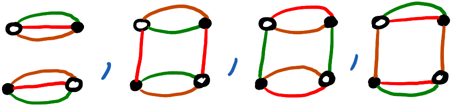
$\{$ LU invariants $\} \Longleftrightarrow \{ \mathrm{Tr}_B(\bar{T}, T),\ B$ q-colored bipartite diag. $\}$

$q = 3$

$T \sim$ 

$\bar{T} \sim$ 

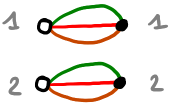
$\underline{n = 1}$ $\{$  $\}$

$\underline{n = 2}$ $\{$  $\}$

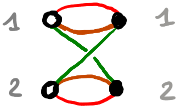
Factorial growth of $\#\{$invariants$\}$ with n.

[Ben Geloun, Ramgoolam]

Parametrization in terms of permutations.



$$\vec{\tau} = (\mathrm{id}, \mathrm{id}, \mathrm{id}) \in S_2^{\times 3}$$



$$\vec{\tau} = ((12), \mathrm{id}, \mathrm{id}) \in S_2^{\times 3}$$

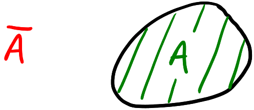
B with $2n$ nodes and q colors $\longleftrightarrow$ $\vec{\tau} = (\tau_1, \dots, \tau_q) \in S_n^{\times q}$ up to $\tau_i \to \rho\, \tau_i\, \nu$

$$\mathrm{Tr}_B(\bar{T}, T) = \mathrm{Tr}_{\vec{\tau}}(\bar{T}, T)$$

$\boxed{\text{Q}}$ Which of those many invariants are most relevant to characterize the multipartite entanglement structure of many-body quantum systems?

② Entanglement and geometry.

Many-body physics. ground states of gapped local Hamiltonians tend to be highly atypical



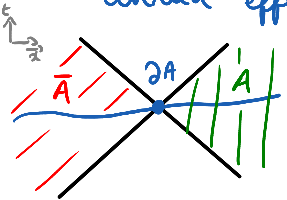
$$\boxed{S_{VN}(\rho_A) \sim cst \times |\partial A|}$$

Area law

$\rightsquigarrow$ Tensor Networks variational Ansätze with a polynomial # of parameters in system size, which obey area laws by construction.

Relativistic QFT.

Unruh effect / Bisognano-Wichman thm.



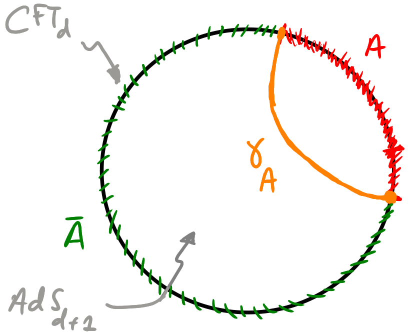
$$\rho_A = \frac{e^{-\beta H_A}}{Z} \quad \text{with} \quad \begin{cases} \beta = 2\pi \\ H_A \ \text{boost Hamiltonian} \end{cases}$$

Universal divergence in $S_{vN}(\rho_A) \propto \text{Area}(\partial A)$

$\rightsquigarrow$ Heuristic derivations of GR [Jacobson '95, '15]

$$\left\{ \begin{array}{l} \delta S(\rho_A) = \underbrace{\eta}_{\text{universal, finite}} \, \delta \text{Area}(\partial A) \\ \text{and} \\ \text{"entanglement equilibrium hypothesis"} \end{array} \right. \iff \text{Einstein's eqs.}$$

$\rightsquigarrow$ Holographic version (AdS/CFT).



Ryu-Takayanagi (RT)

$$\boxed{S_{vN}(A) = \frac{\text{Area}(\gamma_A)}{4G}}$$

γ_A : extremal surface.

RT formula $\Rightarrow$ Einstein's eqs. up to 2^{nd} order

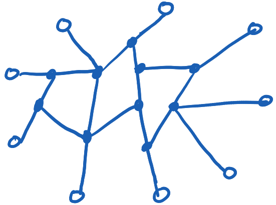
[Faulkner et al. 2017]

Conclusion. Bipartite entanglement and area laws play a central role in many-body physics, relativistic QFT and QG.

Question. Does this interplay between geometry and entanglement extends to the multipartite setting?

$\rightsquigarrow$ Investigate this question on simple toy-models: Random Tensor Networks

③ RTNs [Hayden et al '16]



Graph $G = (V, E)$ with:

$V = V_{bulk} \sqcup V_{\partial}$

$\{\bullet\}$ $\{\circ\}$

Associate Hilbert space $\mathcal{H}_{e,x}$ to each half-edge (e,x)

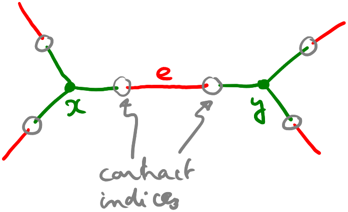
$$\boxed{\dim \mathcal{H}_{e,x} = D}$$

"bond dimension" of network.

Edge data: $|\phi\rangle = \bigotimes_{e=(x,y)\in E} \frac{1}{\sqrt{D}} \sum_{i=1}^{D} |i\rangle_x \otimes |i\rangle_y$

Vertex data: $|\psi\rangle = \bigotimes_{x\in V_b} |\psi\rangle_x$ with $|\psi\rangle_x \in \bigotimes_{e \text{ incident to } x} \mathcal{H}_{e,x}$

Tensor Network state :



$$\boxed{|TN\rangle = \left(\mathbb{1}_{V_\partial} \otimes \langle\psi|\right)|\phi\rangle}$$

Leaves on the boundary V_∂

Random TN: take $\{|\psi\rangle_x\}$ to be indep., Haar-distributed random vectors i.e.

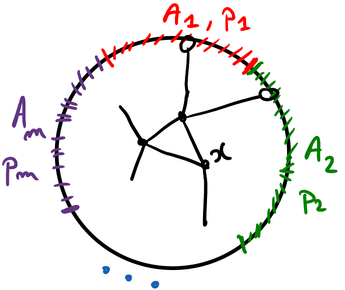
$$\mathbb{E}\left[|\psi\rangle_x {}_x\langle\psi|^{\otimes n}\right] = \int_{U(\dim \mathcal{H}_x)} dU \left[U^\dagger |0\rangle\langle 0| U\right]^{\otimes n}$$

arbitrary

$$= \frac{1}{D(D+1)\cdots(D+n-1)} \sum_{\sigma\in S_n} R_x(\sigma)$$

Permutation operator acting on $\mathcal{H}_x^{\otimes n}$

Trace observables $\longleftrightarrow$ classical spin models



$$G_{\{A_\ell, p_\ell\}} := \mathbb{E}\,\mathrm{Tr}_{p_1,\dots,p_m}\left(|RTN\rangle\langle RTN|\right)$$

$p_\ell \in S_n$

$$= \mathbb{E}\,\langle RTN|^{\otimes n}\, R_{A_1}(p_1)\otimes\cdots\otimes R_{A_m}(p_m)\,|RTN\rangle^{\otimes n}$$

$$G_{\{A_\ell, p_\ell\}} = \sum_{\substack{\sigma : V \to S_n \\ \text{s.t. } \sigma(x) = p_\ell \\ \forall x \in A_\ell}} e^{-\ln D \times E(\sigma)}$$

where $$E(\sigma) = \sum_{e=(x,y)\in E} \overbrace{d(\sigma(x), \sigma(y))}^{\text{Cayley distance on } S_n}$$

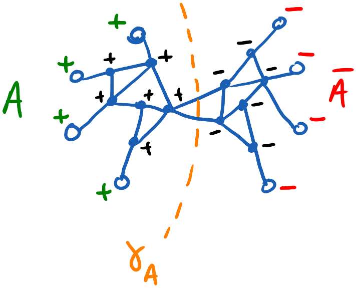
[Haydn et al. '16]

Generalized spin model at inverse temp. $\beta = \ln D$

$$D \to \infty \iff \beta \to 0$$

Hence: at leading order, $\mathcal{O}_{\{A_k, p_k\}}$ is determined by ground states of the model.

Ex. $\mathbb{E}\,\mathrm{Tr}\,\rho_A^2$ $(m = 2,\ n = 2)$ $\quad S_2 = \{\underset{+}{id}, \underset{-}{(12)}\}$



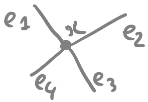
$$\mathbb{E}\,\mathrm{tr}\,\rho_A^2 = \sum_{\sigma : V_b \to \{+,-\}} e^{-\ln D \times E_{Ising}(\sigma)}$$

$$= \mathcal{N}_{g.s.}\, e^{-\ln D\, |\gamma_A|} \left(1 + O\left(\frac{1}{D}\right)\right)$$

$\Rightarrow$ Area law $\quad \boxed{\mathbb{E}\, S_2(\rho_A) \approx |\gamma_A| \ln D}$

④ RTNs with LU-average [w.i.p. with L. Lionni]

Idea. Keep local entanglement structure at each vertex fixed i.e. average over LU instead of full unitary group of $\mathcal{H}_x$.

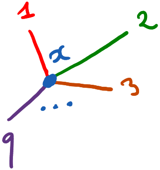


$$\mathcal{H}_x = \bigotimes_{c=1}^{4} \mathcal{H}_{e,x}$$

Hopes:

- Identify which entanglement structures are responsible for area laws.
- Obtain richer entanglement spectra than in the standard case (non-flat spectrum).

Seed state on a vertex:



$$|T\rangle_x = \sum_{i_1,\dots,i_q} \underbrace{T_{i_1\dots i_q}}_{\text{Fixed tensor}} \, |i_1\rangle \otimes \dots \otimes |i_q\rangle$$

Average over $U(D)^{\otimes q} \subsetneq U(Dq)$:

$|\psi\rangle_x \sim$ random state in equivalence class of states with same entanglement structure as $|T\rangle_x$

$$\mathbb{E}\left[|\psi\rangle_x {}_x\langle\psi|^{\otimes n}\right] := \int_{U(D)^{\otimes q}} dU^{(1)} \dots dU^{(q)} \left[\left(\bigotimes_{c=1}^{q} U^{(c)\dagger}\right) \underbrace{|T\rangle_x {}_x\langle T|}_{\text{seed}} \left(\bigotimes_{c=1}^{q} U^{(c)}\right) \right]^{\otimes n}$$

LU average

Weingarten calculus [Collins et al. '00s]

$\rightsquigarrow$ Explicit evaluation of moments

$$\mathbb{E}\left[|\psi\rangle_x {}_x\langle\psi|^{\otimes n}\right] = \sum_{\substack{\sigma_1,\dots,\sigma_q \\ \in S_n}} \underbrace{F_T(\vec{\sigma})}_{\text{State-dep. weight}} \underbrace{\hat{I}(\vec{\sigma})}_{\substack{\text{Pattern of } \delta\text{-contractions} \\ \text{(stranded graph)}}}$$

with

$$\begin{cases} \hat{I}(\vec{\sigma})_{\{i_s^c, j_s^c\}} = \prod_{c=1}^{q} \delta_{i_s^c, j^c_{\sigma_c(s)}} \\ F_T(\vec{\sigma}) = \sum_{\vec{\tau} \in S_n} \underbrace{\mathrm{Tr}_{\vec{\tau}}(\bar{T}, T)}_{\substack{\text{Entanglement} \\ \text{invariant (bubble)}}} \prod_{c=1}^{q} \underbrace{W^{(D)}(\sigma_c \tau_c^{-1})}_{\substack{\text{Weingarten} \\ \text{function}}} \end{cases}$$

Asymptotics of Weingarten's function:

$$D^{n} W^{(D)}(\sigma \tau^{-1}) = D^{-d(\sigma,\tau)} \, M(\sigma\tau^{-1}) \left(1 + O(1/D^2)\right)$$

Cayley distance

Moebius function (rational number)

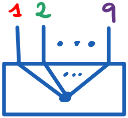
For some family of entangled seed states, this yields:

state-dependent correction

$$F_T(\vec{\sigma}) = \sum_{\vec{\tau}} D^{-\underbrace{(n-\overbrace{C}^{\#\{\text{connected comp. of } B_{\vec{\tau}}\}})(\vec{\tau})}_{\text{minimal when } \tau_1=\tau_2=\cdots=\tau_q} - \underbrace{\sum_{c=1}^{q} d(\sigma_c,\tau_c)}_{\text{minimal when } \vec{\sigma}=\vec{\tau}} - \omega_T(\vec{\tau})} \left(\prod_{c=1}^{q} M(\sigma_c\tau_c^{-1})\right) \times \left(1 + O(1/D^2)\right)$$

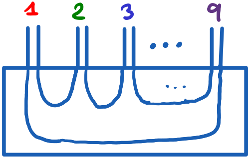
Examples of seed states:

① GHZ state: $|T\rangle = D^{(q-1)/2} \sum_{i=1}^{D} |i\rangle_1 \otimes \cdots \otimes |i\rangle_q$



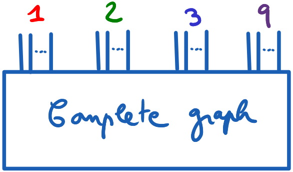
$$\boxed{\omega_T(\vec{\tau}) = 0}$$

② "Cyclic" state: $|T\rangle = D^{q/4}$



$$\boxed{\omega_T(\vec{\tau}) = g_J(B\vec{\tau})}$$ ← genus of Jacket $(1\,2\,3 \cdots q)$

③ "Complete graph" state: $|T\rangle = D^{q/4}$



$$\boxed{\omega_T(\vec{\tau}) = \frac{1}{q-1}\, \omega_{\text{Gurau}}(B\vec{\tau})}$$

Rényi-n entropy $\longleftrightarrow$ generalized "spin" model (S_n)

- $\begin{cases} \tau_1 = \tau_2 = \cdots = \tau_q \\ \vec{\sigma} = \vec{\tau} \end{cases}$ $\rightsquigarrow$ Ising configurations

($n=2$)

- Other contributions $\rightsquigarrow$ Vertex defects

($n=2$)

$s=1$ $\quad$ $s=2$

$\lfloor q/2 \rfloor$ types

Ex. GHZ, $n=2$. Energy functional:

$$E = E_{Ising} + 2\nu_1 + \sum_{s=2}^{\lfloor q/2 \rfloor} \nu_s \quad \text{with} \quad \nu_s = \#\{\text{defects of type } s\}$$

$\rightsquigarrow$ $\boxed{\mathbb{E}S_2(\rho_A) = c_A \ln D}$ with $\boxed{c_A \leq |\gamma_A|}$ $\left\{ \begin{array}{l} c_A < |\gamma_A| \\ \Rightarrow \text{non-flat spectrum.} \end{array} \right.$

Conclusion.

- RTNs with reduced amount of averaging can be analyzed thanks to Weingarten calculus.
- Allows tunable local entanglement structure at each vertex of the network. ⊛
- Rényi entropy evaluation maps to generalized spin model: permutations assigned to half-edges, with ⊛ determining interaction potential at a vertex.
- Interesting new features e.g. non-flat entanglement spectra.

... w.i.p ... stay tuned!